Graded Embeddings of Finite Dimensional Simple Graded Algebras

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Abstract

Let A, B be finite dimensional G-graded algebras over an algebraically closed field $\mathbb{K}$ with $char(\mathbb{K}) = 0$, where G is an abelian group, and let $Id_G(A)$ be the set of graded identities of A (res. $Id_G(B)$). We show that if A, B are G-simple then there is a graded embedding $\phi : A \to B$ iff $Id_G(B) \subseteq Id_G(A)$. We also give a weaker generalization for the case where A is G-semisimple and B is arbitrary.

1 Introduction

Let A be an algebra and G be a group. A G-grading of A is a decomposition $A = \bigoplus_{g\in G} A_g$ where $A_g A_h \subseteq A_{gh}$. A polynomial $p(x_{g_1,1}, ..., x_{g_n,n})$ with non-commutative graded indeterminates is called a graded identity of A if $p(a_1, ..., a_n) = 0$ for any assignment in A such that $a_i \in A_{g_i}$, and we define $Id_G(A)$ to be the set of the graded identities of A.

If A and B are isomorphic as G-graded algebras then obviously $Id_G(A) = Id_G(B)$, and we can ask if the converse is also true. In the current setting this is not true. For example, for any algebra A we have $Id_G(A) = Id_G(A \oplus A)$ while in general A isn't isomorphic to $A \oplus A$. Another less trivial but important example comes from scalar extension. It is well-known that if A, B are algebras over a field $\mathbb{F}$ with $char(\mathbb{F}) = 0$, $\mathbb{K}$ is a field extension of $\mathbb{F}$ and $A \otimes_{\mathbb{F}} \mathbb{K} \cong_G B \otimes_{\mathbb{F}} \mathbb{K}$, then $Id_G(A) = Id_G(B)$, even though A doesn't have to be isomorphic to B. For example the quaternion algebra $\mathbb{H}_{\mathbb{R}}$ and the matrix algebra $M_2(\mathbb{R})$ are not isomorphic, while $\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}) \cong M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ so $Id(\mathbb{H}_{\mathbb{R}}) = Id(M_2(\mathbb{R}))$.

To avoid such counter examples as above, we restrict ourselves to algebraically closed fields, and also assume that A and B are finite dimensional G-simple algebras. In this case, it is true that if $Id_G(A) = Id_G(B)$ then $A \cong_G B$. This was first proved for G abelian and the order of any finite subgroup of G is invertible in $\mathbb{K}$ (Koshlukov and Zaicev [1]) and was later proved for G arbitrary and $char(\mathbb{K}) = 0$ (Aljadeff and Haile [2]).

These two results show that in order to identify a finite dimensional G-simple algebra, all we need to know is its graded identities. Seeing that the equality $Id_G(A) = Id_G(B)$ means that $A \cong_G B$ in these cases, we can ask what other relations between two G-graded algebras A and B can be derived from relations between their respective ideals of identities.

If $A \subseteq B$ as graded algebras, then any polynomial identity of B is certainly an identity of A so $Id_G(B) \subseteq Id_G(A)$, so here too we can ask whether or not the converse is true.

Suppose now that we are in the non-graded case, and we have $Id(B) \subseteq Id(A)$. If A, B are finite dimensional simple algebras over an algebraically closed field $\mathbb{K}$, then by Artin-Wedderburn theorem we know that $A = M_n(\mathbb{K})$, $B = M_m(\mathbb{K})$ for some $m, n \in \mathbb{N}$. Let $St_r(x_1, ..., x_r) = \sum_{\sigma \in S_r} (-1)^\sigma \prod x_{\sigma(i)}$ be the standard polynomial with r indeterminates, then by Amitsur-Levitzki theorem $St_r \in Id(M_n(\mathbb{K}))$ iff $r \geq 2n$. Using the Amitsur-Levitzki theorem we first get that $St_{2m} \in Id(B)$ so $St_{2m} \in Id(A)$, and then using it a second time we get that $2m \geq 2n \Rightarrow m \geq n$. Since $m \geq n$ we can find an injective map $\varphi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$, so in the non-graded case, for two simple finite dimensional algebras A, B there is an embedding $\varphi : A \to B$ iff there is an inclusion $Id(B) \subseteq Id(A)$.

The matrix algebras $M_n(\mathbb{K})$, as in the example above, together with $M_n(E)$ and $M_{n,m}(E)$, where E is the infinite Grassmann algebra, play a very important part in the classical PI theory of non-graded algebras. For a T ideal I of identities and an algebra A we denote by $I(A)$ the ideal of all the evaluations of polynomials in I with elements in A. In [3] Kemer showed that for any PI algebra A of characteristic zero there is a T ideal I such that ${}^{A}/_{I(A)}$ is PI equivalent to a direct sum of verbally prime algebras, and every verbally prime algebra is PI equivalent to one of $M_n(\mathbb{K})$, $M_n(E)$ or $M_{n,m}(E)$. The question, if $Id(A) \subseteq Id(B)$ implies $B \subseteq A$, for these three types of algebras, was answered positively by Berele in [4], with a slight exception in the case of $M_{n,m}(E)$.

As we showed above, the proof for non-graded finite dimensional simple algebras, or in other words, matrix algebras, is fairly simple. In the graded case the situation is much more complicated. Each matrix algebra $M_n(\mathbb{K})$ may have many non-isomorphic G-gradings, and so in particular they are G-simple, and there are also other graded algebras which are not isomorphic (non-graded) to matrix algebras, but still G-simple. In this paper, we show that the claim is true for G-simple finite dimensional algebras if the group G is abelian.

Theorem 1. *Let G be an abelian group, and $\mathbb{K}$ an algebraically closed field with $char(\mathbb{K}) = 0$. Let A, B be two G-graded, finite dimensional algebras over $\mathbb{K}$.*

1. *If A, B are G-simple then there is a graded embedding $A \hookrightarrow_G B$ iff $Id_G(B) \subseteq Id_G(A)$.*
2. *If A is G-semisimple, G is finite and B has a unit then $Id_G(B) \subseteq Id_G(A)$ iff there is a graded embedding $A \hookrightarrow_G B^N$ for N large enough.*
3. *If A is simple then we can choose $N = 1$, so $Id_G(B) \subseteq Id_G(A)$ iff there is a graded embedding $A \hookrightarrow_G B$.*

We also give a special case for G arbitrary. Let $(g_1, ..., g_n) \in G^n$ be a tuple of elements in G, then we can define a G-grading on $M_n(\mathbb{K})$ by setting $\deg(E_{i,j}) = g_i^{-1} g_j$ where $E_{i,j}$ is the zero matrix with 1 in the (i, j) location. We call such a grading an elementary grading.

Theorem 2. *Let G be an arbitrary group and $\mathbb{K}$ an algebraically closed field with $char(\mathbb{K}) = 0$. Let A be a finite dimensional G-simple algebra, and $B = M_n(\mathbb{K})$ with an elementary G-grading, then $Id_G(B) \subseteq Id_G(A)$ iff there is a graded embedding $A \hookrightarrow_G B$.*

Let $\mathbb{K}$ be any field with $char(\mathbb{K}) = 0$ and $\overline{\mathbb{K}}$ its algebraic closure. As we mentioned before, a trivial reason for two graded algebras A, B over $\mathbb{K}$ to have the same graded

identities is that $A \otimes_{\mathbb{K}} \overline{\mathbb{K}} \cong_G B \otimes_{\mathbb{K}} \overline{\mathbb{K}}$, which means that B is a twisted form of A in the language of Galois descent. Let A be an algebra over $\mathbb{K}$ and suppose that $\bar{A} = A \otimes_{\mathbb{K}} \bar{\mathbb{K}}$ is G-simple over $\overline{\mathbb{K}}$ and we want to build a generic algebra A_{gen} that specialize exactly to all the twisted forms of A. If R_1 is a graded algebra and R_2 is a homomorphic image of R_1 then clearly $Id_G(R_1) \subseteq Id_G(R_2)$, so in our case we will have $Id_G(A_{gen}) \subseteq Id_G(B)$ for any homomorphic image B of A_{gen}, and in particular $Id_G(A_{gen}) \subseteq Id_G(A)$.

With this in mind, let $\mathbb{K}\langle X_G\rangle$ be the free algebra with noncommutative indeterminates $\{x_{g,i}\}_{g\in G, i\in\mathbb{N}}$ and let $\tilde{A} = {}^{\mathbb{K}\langle X_G\rangle}/_{Id_G(A)}$, then in particular we have $Id_G(A) = Id_G(\tilde{A})$. It is clear that any finitely generated algebra B with $Id_G(A) \subseteq Id_G(B)$ is a homomorphic image of $\tilde{A}$, and in particular any twisted form of A. The problem is that $\tilde{A}$ can have other homomorphic images that are not twisted forms (for example the trivial algebra $\{0\}$). Let B be a specialization of $\tilde{A}$, set $\bar{B} = B \otimes_{\mathbb{K}} \bar{\mathbb{K}}$ and assume that $\bar{B}$ is finite dimensional simple, then the theorem above shows that though we don't necessarily have $\bar{B} \cong_G \overline{A}$, we do know that $\bar{B} \hookrightarrow_G \overline{A}$. This shows that a good place to start to look for A_{gen} is with the relative free algebra over $Id_G(A)$. Such constructions were already carried for twisted group algebras in [5](definition inside the paper).

In the second part of theorem 1, we see that if A and B are G-semisimple, then $Id_G(B) \subseteq Id_G(A)$ doesn't have to come from graded embedding $A \hookrightarrow_G B$. The situation becomes much simpler when we consider only equality of the ideals, and combine it with the results from [1, 2] about isomorphisms of G-simple algebras.

The next lemma was proved by Berele in [4] for the case of verbally prime algebras, and we give here an easy proof for the case of G-simple algebras. We say that a set $\{A_i\}_1^n$ of graded algebras is minimal if for all j we have $\bigcap_k Id_G(A_k) \subsetneqq \bigcap_{k\neq j} Id_G(A_k)$.

Lemma 3. *Let $\mathbb{K}$ be an arbitrary field. Let $\{A_i\}_1^n, \{B_i\}_1^m$ be two minimal sets of G-simple algebras. If $A = \bigoplus_1^n A_i$, $B = \bigoplus_1^m B_j$ and $Id_G(A) = Id_G(B)$ then $m = n$, and there is a permutation $\tau : [n] \to [n]$ such that $Id_G(A_i) = Id_G(B_{\tau(i)})$.*

If we also know that $Id_G(A_i) = Id_G(B_j)$ implies that $A_i \cong_G B_j$ then the lemma above shows that $Id_G(A) = Id_G(B)$ implies that $A \cong_G B$. We now use the results from[1, 2] to conclude that:

Corollary 4. *Let $\{A_i\}_1^n, \{B_j\}_1^m$ be minimal sets of G-simple finite dimensional algebras and let $A = \bigoplus_1^n A_i$, $B = \bigoplus_1^m B_j$. Suppose that*

1. *G is abelian and for every finite subgroup $H \leq G$ we have $|H|^{-1} \in \mathbb{K}$*
2. *or G is arbitrary and $char(\mathbb{K}) = 0$*

then $Id_G(A) = Id_G(B)$ iff $A \cong_G B$.

2 Preliminaries

2.1 Graded Algebras

We start with some basic definitions.

Let A be an algebra over a field $\mathbb{K}$, and let G be a group. A *G-grading* of A is a decomposition of A as a vector space over $\mathbb{K}$ to $\bigoplus_{g\in G} A_g$, such that $A_g \cdot A_h \subseteq A_{gh}$ for every $g,h \in G$. A sub-algebra (respectively left, right or two sided ideal) $B \subseteq A$ is called a *graded sub-algebra* if $B = \bigoplus\limits_{g\in G}(B \cap A_g)$. The algebra A is called *G-simple* if it has no non-trivial two sided graded ideals and $A \cdot A \neq 0$.

Example 5 (Twisted group algebra)**.** Let G be a finite group and $\mathbb{K}$ a field. A function $\alpha : G \times G \to \mathbb{K}^\times$ is said to be a 2-cocycle of G (denoted by $\alpha \in Z^2(G, \mathbb{K}^\times)$) if it satisfies

$$\forall u, v, w \in G \qquad \alpha(u, v)\alpha(uv, w) = \alpha(u, vw)\alpha(v, w)$$

Let $A = \mathbb{K}^\alpha G$ be the algebra such that as a vector space it is the direct sum $\bigoplus\limits_{g\in G}\mathbb{K}U_g$ and the product is defined by $U_gU_h = \alpha(g, h)U_{gh}$. The property of α above shows that $\mathbb{K}^\alpha G$ is an associative algebra, and setting $A_g = \mathbb{K}U_g$ induces a grading on $\mathbb{K}^\alpha G$. Taking a new basis $V_g = \alpha(e, e)^{-1}U_g$ we get

$$\begin{aligned} \alpha(g, e)\alpha(ge, e) &= \alpha(g, ee)\alpha(e, e) \quad \Rightarrow \quad \alpha(g, e) = \alpha(e, e) \\ V_gV_e &= \alpha(e, e)^{-2}U_gU_e = \alpha(e, e)^{-2}\alpha(g, e)U_g = \alpha(e, e)^{-1}\alpha(g, e)V_g = V_g \end{aligned}$$

and in a similar way $V_eV_g = V_g$ so V_e is a unit. Because $0 \neq V_gV_{g^{-1}} \in \mathbb{K}V_e$, each graded component A_g is spanned by an invertible element, and therefore each nonzero graded ideal of A must be all A, or in other words $\mathbb{K}^\alpha G$ is G-simple.

For a tuple $\bar{g} \in G^n$ we denote by $\alpha(\bar{g}) = \alpha(g_1, ..., g_n) \in \mathbb{K}$ the constant such that

$$U_{g_1}U_{g_2}\cdots U_{g_n} = \alpha(g_1, ..., g_n)U_{g_1g_2\cdots g_n}$$

Example 6 (Elementary Grading)**.** Let $\bar{s} \in G^n$. We denote by $M_{\bar{s}}(\mathbb{K})$ the algebra $A = M_n(\mathbb{K})$ with the grading

$$A_g = span\left\{E_{i,j} \;\mid\; \deg(E_{i,j}) = s_i^{-1}s_j = g\right\}.$$

A is simple as an algebra so it must also be simple as a graded algebra.

Let A, B be two G-graded algebras. An algebra homomorphism $\varphi : A \to B$ is called graded if $\varphi(A_g) \subseteq B_g$ for any $g \in G$, and if this is true then $\ker(\varphi)$ is a graded two sided ideal. Notice that if A is simple then φ is either the zero homomorphism, or injective.

A theorem by Bahturin, Sehgal and Zaicev [6] gives us the structure of all G-simple algebras of finite dimension.

Theorem 7. *Let $R = \bigoplus\limits_{g\in G}R_g$ be a G-graded finite-dimensional algebra over an algebraically closed field $\mathbb{K}$. Suppose that $char(\mathbb{K}) = 0$ or $char(\mathbb{K})$ is coprime with the order of each finite subgroup of G. Then R is G-simple iff $R \cong \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$ where H is a finite subgroup of G, $\alpha \in Z^2(H, \mathbb{K}^\times)$, $\bar{s} \in G^n$ and the grading is defined by*

$$R_g = span\left\{U_h \otimes E_{i,j} \;\mid\; s_i^{-1}hs_j = g\right\}$$

2.2 Graded Identities

For a group G we define $X_g = \{x_{g,i} \mid i \in \mathbb{N}\}$, $X_G = \bigcup_{g\in G} X_g$ and $\mathbb{K}\langle X_G\rangle$ to be the free $\mathbb{K}$ algebra generated by the noncommutative indeterminates X_G. For a monomial $f = \prod_{j=1}^{k} x_{g_j,i_j}$ we set the degree $\deg(f) = \prod_1^k g_j$. We define a G grading on $\mathbb{K}\langle X_G\rangle$ by

$$\mathbb{K}\langle X_G\rangle_g = span\{f \in \mathbb{F}\langle X_G\rangle \mid f \text{ is a monomial, } \deg(f) = g\}$$

A *graded assignment* in A for a polynomial $f \in \mathbb{K}\langle X_G\rangle$, is such that each indeterminate $x_{g,i}$ is substituted by an element from A_g. The polynomial f is called a *graded identity* of A if the evaluation of f on any graded assignment is zero, or in other words $f\mid_A = \{0\}$. We define the *ideal of identities* of A by

$$Id_G(A) = \{f \in \mathbb{K}\langle X_G\rangle \mid f\mid_A = \{0\}\}$$

It is easy to check that this is a graded ideal of $\mathbb{K}\langle X_G\rangle$. This ideal is also closed under graded endomorphism, so if $f \in Id_G(A)$, then we can substitute an indeterminate $x_{g,i}$ in f by any polynomial $h \in \mathbb{K}\langle X_G\rangle$ of degree g and the the result will still be in $Id_G(A)$.

A graded ideal in $\mathbb{K}\langle X_G\rangle$ that is closed under graded endomorphism is called a T *ideal.*

A graded polynomial $f(x_{g_1,1}, ..., x_{g_n,n})$ is called *multilinear* if it is linear in each of the indeterminates. Every multilinear polynomial is of the form

$$\sum_{\sigma\in S_n} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)}$$

If S is some spanning set of homogeneous elements in A, then it is easy to check that a multilinear polynomial f is an identity of A iff the result of any graded assignment of S in f is zero. It is well known that for $\mathbb{K}$ with $char(\mathbb{K}) = 0$, the ideal $Id_G(A)$ is generated as a T ideal by multilinear polynomials, so in most of the paper we will assume that $char(\mathbb{K}) = 0$ and only concentrate on multilinear polynomials.

3 Graded Embeddings of a Simple Algebra in a Simple Algebra

In the following section G will be a group, and A, B will be two finite dimensional G-simple algebra over an algebraically closed field $\mathbb{K}$ with $char(\mathbb{K}) = 0$.

The question we ask is if the condition $Id_G(B) \subseteq Id_G(A)$ is enough in order to find a graded embedding $\varphi : A \hookrightarrow B$.

Before we start we give some standard operations on G-simple algebras that return G isomorphic algebras (Lemma 1.3 [2])

Lemma 8. *Let $H \leq G$ be a finite sub group, $\alpha \in Z^2(H, \mathbb{K}^\times)$ and $\bar{s} = (s_1, ..., s_r) \in G^r$.*

1. *If $\sigma \in S_r$ is any permutation and $\bar{s}^\sigma \in G^r$ is defined by $\bar{s}^\sigma_i = s_{\sigma(i)}$ then $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \cong_G \mathbb{K}^\alpha H \otimes M_{\bar{s}^\sigma}(\mathbb{K})$.*

2. *Suppose that $Hs_i = Ht$. Denote by $\bar{s}' = (s_1, ..., s_{i-1}, t, s_{i+1}, ..., s_m)$ then $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(K) \cong_G \mathbb{K}^\alpha H \otimes M_{\bar{s}'}(K)$.*

3. *For any $g \in G$ define $g\bar{s} = (gs_1, ..., gs_n)$, $H_g = gHg^{-1}$ and $\alpha_g \in Z^2(H_g, \mathbb{K}^\times)$ by $\alpha_g(gh_1g^{-1}, gh_2g^{-1}) = \alpha(h_1, h_2)$ then $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \cong_G \mathbb{K}^{\alpha_g} H_g \otimes M_{g\bar{s}}(\mathbb{K})$*

Proof. .

1. Use the graded isomorphism $U_h \otimes E_{i,j} \mapsto U_h \otimes E_{\sigma^{-1}(i),\sigma^{-1}(j)}$.

2. Denote by $\tilde{h} \in H$ the element such that $s_i = \tilde{h}t$ and then the isomorphism $\varphi : K^\beta H \otimes M_{\bar{s}}(K) \to K^\beta H \otimes M_{\bar{s}'}(K)$ is defined by

$$\varphi\left(U_h E_{j,k}\right) = \begin{cases} U_h E_{j,k} & j,k \neq i \\ \beta(\tilde{h}, \tilde{h}^{-1})^{-1} U_{\tilde{h}^{-1}} U_h E_{i,k} & j = i,\ k \neq i \\ U_h U_{\tilde{h}} E_{j,i} & j \neq i,\ k = i \\ \beta(\tilde{h}, \tilde{h}^{-1})^{-1} U_{\tilde{h}^{-1}} U_h U_{\tilde{h}} E_{i,i} & j,k = i \end{cases}$$

3. Use the graded isomorphism $\varphi : U_h \otimes E_{i,j} \mapsto U_{ghg^{-1}} \otimes E_{i,j}$.

□

Parts 1 and 2 of the lemma above shows that we can change the tuple $\bar{s}$ by permuting its elements and by changing elements in the same right H coset, and it doesn't change the algebra.

Definition 9. Let $H \leq G$, and $\bar{s}, \bar{s}' \in G^r$, then we say that $\bar{s}$ and $\bar{s}'$ are equivalent modulo H (or just equivalent) and write $\bar{s} \sim_H \bar{s}'$ if we can get $\bar{s}'$ from $\bar{s}$ by a sequence of permutations, and multiplication from the left of individual components by elements of H. If $\bar{s}$ has a sub tuple that is equivalent to $\bar{s}'$ then we write $\bar{s} \gtrsim_H \bar{s}'$.

Write $\bar{s} \sim \bar{s}'$ (respectively $\bar{s} \gtrsim \bar{s}'$) for $\bar{s} \sim_{\{e\}} \bar{s}'$ (respectively $\bar{s} \gtrsim_{\{e\}} \bar{s}'$).

The lemma above shows that if $\bar{s} \sim_H \bar{s}'$ and α is any 2-cocycle of H then $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \cong_G \mathbb{K}^\alpha H \otimes M_{\bar{s}'}(\mathbb{K})$. If we only have $\bar{s} \succsim_H \bar{s}'$ then we get only a graded embedding $\mathbb{K}^\alpha H \otimes M_{\bar{s}'}(\mathbb{K}) \hookrightarrow \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$.

Let $\bar{s} \in G^{r_1}$ and $\bar{t} \in G^{r_2}$ be two tuples and define the graded algebra $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \otimes M_{\bar{t}}(\mathbb{K})$ to be the algebra $\mathbb{K}^\alpha H \otimes M_{r_1}(\mathbb{K}) \otimes M_{r_2}(\mathbb{K})$ with grading

$$\deg(U_h \otimes E_{i_1,j_1} \otimes E_{i_2,j_2}) = t_{i_2}^{-1} s_{i_1}^{-1} h s_{j_1} t_{j_2}$$

Notice that if G is not abelian then we usually don't have $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \otimes M_{\bar{t}}(\mathbb{K}) \cong_G \mathbb{K}^\alpha H \otimes M_{\bar{t}}(\mathbb{K}) \otimes M_{\bar{s}}(\mathbb{K})$ with the particular important exception that $\bar{s} = (e, e, .., e)$. For these two tuples define $\bar{s} \times \bar{t} \in G^{r_1 r_2}$ to be the tuple such that $(\bar{s} \times \bar{t})_{(i,j)} = s_i t_j$ (up to a permutation) then

$$\mathbb{K}^\alpha H \otimes M_{\bar{s} \times \bar{t}}(\mathbb{K}) \cong_G \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}) \otimes M_{\bar{t}}(\mathbb{K})$$

It is easy to see that if $\bar{s} \sim_H \bar{s}'$ then $\bar{s} \times \bar{t} \sim_H \bar{s}' \times \bar{t}$. If H is normal in G then for each $h \in H,\ s, t \in G$ we have $shs^{-1} \in H$ and $(shs^{-1})st = s(ht)$, and therefore if $\bar{t} \sim_H \bar{t}'$ then $\bar{s} \times \bar{t} \sim_H \bar{s} \times \bar{t}'$.

For $\bar{u} \in G^n,\ \bar{v} \in G^m$ define $(\bar{u}, \bar{v}) = \bar{u} + \bar{v} = (u_1, ..., u_n, v_1, ..., v_m)$ then

$$(\bar{u} + \bar{v}) \times \bar{t} = \bar{u} \times \bar{t} + \bar{v} \times \bar{t} \qquad ; \qquad \bar{t} \times (\bar{u} + \bar{v}) = \bar{t} \times \bar{u} + \bar{t} \times \bar{v}$$

For $d \in \mathbb{N}$ we define $\bar{d}$ to be $\bar{d} := (\overbrace{e, ..., e}^{d\ times})$.

G-Envelope

We now describe a second operation on graded algebras that will be useful in the proof.

Let $E = E_0 \oplus E_1$ be the infinite Grassmann algebra, and $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. Denote by $E \widehat{\otimes} A$ the Grassmann envelope of A, with the $\mathbb{Z}_2$ grading defined by $\left(E \widehat{\otimes} A\right)_0 = E_0 \otimes A_0$ and $\left(E \widehat{\otimes} A\right)_1 = E_1 \otimes A_1$. This is a very important operation in general, and in particular in PI-theory. One of the main reasons for its importance is that if B is another $\mathbb{Z}_2$ graded algebra then $Id_{\mathbb{Z}_2}(A) = Id_{\mathbb{Z}_2}(B)$ iff $Id_{\mathbb{Z}_2}(E \widehat{\otimes} A) = Id_{\mathbb{Z}_2}(E \widehat{\otimes} B)$. We now extend this envelope operation to general groups.

Definition 10 (G-envelope)**.** Let A, B be two G-graded algebras. We denote by $A \widehat{\otimes} B$ to be the G-graded algebra defined by $\left(A \widehat{\otimes} B\right)_g = A_g \otimes B_g$.

Let $A = \mathbb{K}^\alpha G$ be some twisted group algebra and B a G-graded algebra and denote $B^\alpha = A \widehat{\otimes} B$ and call this algebra the α envelope of B. We claim that this operation enjoys the same properties as the Grassmann envelope.

Let $\bar{g} = (g_1, ..., g_n) \in G^n$, and recall that $\alpha(\bar{g})$ is the scalar such that $\prod U_{g_i} = \alpha(\bar{g}) U_{\prod g_i}$. For $\sigma \in S_n$ denote $\bar{g}^\sigma = (g_{\sigma(1)}, ..., g_{\sigma(n)})$. For a tuple $\bar{g} \in G^n,\ g \in G$, define $S_n^{\bar{g},g} = \left\{\sigma \in S_n \mid \prod g_{\sigma(i)} = g\right\}$, then any multilinear polynomial in the indeterminates $(x_{g_i,i})_1^n$ which is homogeneous of degree g is of the form $f(x_{g_1,1}, ..., x_{g_n,n}) = \sum_{\sigma \in S_n^{\bar{g},g}} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)}$. Notice that if G is abelian then $S_n^{\bar{g},g} = S_n$ whenever $g = \prod g_i$, and

is empty otherwise. The polynomial f is an identity of B^α iff for all $b_i \in B_{g_i}$ we have

$$\begin{aligned} 0 &= \sum_\sigma \lambda_\sigma \prod_i \left(U_{g_{\sigma(i)}} \otimes b_i\right) = U_g \otimes \sum_\sigma \lambda_\sigma \alpha(\bar{g}^\sigma) \prod_i b_{\sigma(i)} \\ \iff & \quad 0 = \sum_\sigma \lambda_\sigma \alpha(\bar{g}^\sigma) \prod_i b_{\sigma(i)} \end{aligned}$$

Set $f^\alpha = \sum_\sigma \lambda_\sigma \alpha(\bar{g}^\sigma) \prod_i x_{\sigma(i)}$ then we just showed that $f \in Id_G(B^\alpha)$ iff $f^\alpha \in Id_G(B)$.

Lemma 11. *Let B_1, B_2 be two G-graded algebra and $\alpha \in Z^2(G, \mathbb{K}^\times)$ then*

1. *$(B^\alpha)^{\alpha^{-1}} \cong_G B$.*
2. *There is a graded embedding $B_1 \hookrightarrow_G B_2$ iff there is a graded embedding $B_1^\alpha \hookrightarrow_G B_2^\alpha$.*
3. *There is an inclusion $Id_G(B_1) \subseteq Id_G(B_2)$ iff $Id_G(B_1^\alpha) \subseteq Id_G(B_2^\alpha)$.*

Proof. .

1. Write $\mathbb{K}^\alpha G = \bigoplus \mathbb{K} U_g$ and $\mathbb{K}^{\alpha^{-1}} G = \bigoplus \mathbb{K} V_g$, then $\psi : (B^\alpha)^{\alpha^{-1}} \to B$ defined by $\psi(V_g \otimes U_g \otimes b) = b$ for $b \in B_g$ is a graded isomorphism.
2. If $\varphi : B_1 \to B_2$ then $\varphi^\alpha : B_1^\alpha \to B_2^\alpha$ defined by $\varphi^\alpha(U_g \otimes b) = U_g \otimes \varphi(b)$ for $b \in B_g$ is a graded embedding. The other direction is proved using part (1).
3. Assume that $f \in Id_G(B_1^\alpha)$ then $f^\alpha \in Id_G(B_1) \subseteq Id_G(B_2)$ so $f \in Id_G(B_2^\alpha)$.

□

A standard proof shows that if B is G-simple then B^α is also G-simple. In the case where B is finite dimensional we know exactly what is B^α.

Theorem 12. *Let $B = \mathbb{K}^\beta H \otimes M_{\bar{s}}(\mathbb{K})$ then $B^\alpha \cong_G \mathbb{K}^{\beta\cdot\alpha} H \otimes M_{\bar{s}}(\mathbb{K})$.*

Proof. Let $\mathbb{K}^\alpha G = \bigoplus \mathbb{K} U_g$, $\mathbb{K}^\beta H = \bigoplus \mathbb{K} V_h$ and $\mathbb{K}^{\beta\cdot\alpha} H = \bigoplus \mathbb{K} W_h$. We define $\psi : B^\alpha \to \mathbb{K}^{\beta\cdot\alpha} G \otimes M_{\bar{s}}(\mathbb{K})$ by

$$\psi(U_{s_i^{-1} g s_j} \otimes V_g \otimes E_{i,j}) = W_g \otimes \frac{\sqrt{\alpha(s_i^{-1}, s_i)\alpha(s_j^{-1}, s_j)}}{\alpha(s_i^{-1}, g, s_j)} E_{i,j}$$

We leave the interested reader to show that this map multiplicative. □

Let E be a trivially graded algebra then the algebra $\mathbb{K}^\alpha G \otimes E$ has a natural G-grading. Similar to the G-envelope, one can show a connection between E and graded algebra $\mathbb{K}^\alpha G \otimes E$.

Lemma 13. *Let E be trivially graded algebras, and $\alpha \in Z^2(G, \mathbb{K}^\times)$. Let $\bar{g} = (g_1, ..., g_n) \in G^n$, $g \in G$ then $f(x_{g_1}, ..., x_{g_n}) = \sum_{\sigma \in S_n^{\bar{g}, g}} \lambda_\sigma \prod x_{g_{\sigma(i)}, \sigma(i)}$ is a graded identity of $\mathbb{K}^\alpha G \otimes E$ iff*

$$f^{\bar{g},\alpha}(x_1, ..., x_n) := \sum_{\sigma \in S_n^{\bar{g}, g}} \lambda_\sigma \alpha(\bar{g}^\sigma) \prod x_{\sigma(i)} \in Id(E)$$

Proof. Similar to the proof with α envelope. □

3.1 Part 1 - $A = \mathbb{K}^{\alpha}H \otimes M_{\bar{s}}(\mathbb{K})$, $B = \mathbb{K}^{\beta}G \otimes M_{r_2}(\mathbb{K})$, $H \leq G$ and $\bar{s} \in G^{r_1}$

We now start to build the graded embeddings. In all the following steps, we assume that G is abelian and $\mathbb{K}$ is an algebraically closed field of characteristic zero. We will always have a twisted group algebra $\mathbb{K}^{\alpha}G_1$ in A and $\mathbb{K}^{\beta}G_2$ in B, and we denote the basis of $\mathbb{K}^{\alpha}G_1$ with V_g and the basis for $\mathbb{K}^{\beta}G_2$ with U_g for $g \in G$.

Theorem 14. *Let G be a finite abelian group, $H \leq G$ a subgroup, $\bar{s} \in G^{r_1}$ a tuple and $\alpha \in Z^2(H, \mathbb{K}^{\times})$, $\beta \in Z^2(G, \mathbb{K}^{\times})$. Let $\beta' = \beta\mid_H$, and let d be the dimension of the smallest representation of $\mathbb{K}^{\alpha/\beta'}H$ then for $A = \mathbb{K}^{\alpha}H \otimes M_{\bar{s}}(\mathbb{K})$ and $B = \mathbb{K}^{\beta}G \otimes M_{r_2}(\mathbb{K})$ the following are equal*

1. *There is a graded embedding $A \hookrightarrow B$*
2. *$Id_G(B) \subseteq Id_G(A)$*
3. *$r_2 \geq d \cdot |\bar{s}| = d \cdot r_1$, or equivalently $\overline{r_2} \gtrsim_G \bar{d} \times \bar{s}$*

Taking the β^{-1} envelope on both algebras we can use Lemma 11 to see that part (1) is equivalent to a graded embedding $A^{\beta^{-1}} \hookrightarrow B^{\beta^{-1}}$ and part (2) is equivalent to $Id_G(B^{\beta^{-1}}) \subseteq Id_G(A^{\beta^{-1}})$. By Theorem 12 we get that $B^{\beta^{-1}} \cong_G \mathbb{K}G \otimes M_{r_2}(\mathbb{K})$ and $A^{\beta^{-1}} \cong_G \mathbb{K}^{\alpha/\beta}H \otimes M_{\bar{s}}(\mathbb{K})$, so part (3) remains without change. This gives us a reduction to the problem where the cocycle in B is trivial, and from now on we assume that $\beta = 1$.

Assume first that $A = \mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K})$, so the matrix part of A is trivially graded. β is trivial so we need a way to map the possibly non-trivial cocycle α to B, but since β is trivial then we will have to compensate using the matrix part of B. In order to prove the theorem we first give a lemma that shows the compensation costs a $d \times d$ matrix algebra in B where d is the dimension of the smallest representation of $\mathbb{K}^{\alpha}H$.

Lemma 15. *Let G be an arbitrary finite group, $r_1, r_2 \in \mathbb{N}$, $\alpha \in Z^2(G, \mathbb{K}^{\times})$, and denote by d the dimension of the smallest representation of $\mathbb{K}^{\alpha}G$, then*

1. *There is a graded embedding $\mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K}) \hookrightarrow \mathbb{K}G \otimes M_{r_2}(\mathbb{K})$ iff $r_2 \geq r_1 d$.*
2. *If G is abelian and $Id_G(\mathbb{K}G \otimes M_{r_2}(\mathbb{K})) \subseteq Id_G(\mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K}))$ then $r_2 \geq r_1 d$.*

Proof. 1. Denote by $\epsilon : \mathbb{K}G \to \mathbb{K}$ the augmentation representation and by $\rho : \mathbb{K}^{\alpha}G \to M_d(\mathbb{K})$ the smallest representation of $\mathbb{K}^{\alpha}G$.
Suppose first that there is a graded embedding $\varphi : \mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K}) \hookrightarrow \mathbb{K}G \otimes M_{r_2}(\mathbb{K})$ and compose it with $\epsilon \otimes id$, then we have a map $\mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K}) \to M_{r_2}(\mathbb{K})$. The function φ is a graded embedding so $\varphi(V_e \otimes I) = U_e \otimes a$ for some $0 \neq a \in M_{r_2}(\mathbb{K})$ and therefore $\epsilon \circ \varphi(V_e \otimes I) = \epsilon(U_e) \otimes a = a \neq 0$ so $\epsilon \circ \varphi \neq 0$. Decompose $\mathbb{K}^{\alpha}G$ to a direct sum of matrix algebras of dimensions $d_1^2 \leq d_2^2 \leq \cdots \leq d_k^2$ where $d_1 = d$. Since $\epsilon \circ \varphi \neq 0$, at least one of these matrix algebras, which are simple, is mapped injectively into $M_{r_2}(\mathbb{K})$ so there is $1 \leq i \leq k$ such that $d^2 r_1^2 \leq d_i^2 r_1^2 \leq r_2^2$ and therefore $dr_1 \leq r_2$.
Suppose now that $dr_1 \leq r_2$, then we have an embedding $\psi : M_d(\mathbb{K}) \otimes M_{r_1}(\mathbb{K}) \to M_{r_2}(\mathbb{K})$. Define $\varphi : \mathbb{K}^{\alpha}G \otimes M_{r_1}(\mathbb{K}) \to \mathbb{K}G \otimes M_{r_2}(\mathbb{K})$ by

$$\varphi(\sum_{g \in G} V_g \otimes a_g) = \sum_{g \in G} U_g \otimes \psi(\rho(V_g) \otimes a_g)$$

then φ is linear, but also multiplicative since

$$\begin{aligned}\varphi\left(\left(V_g\otimes a_g\right)\left(V_h\otimes b_h\right)\right) &= \varphi\left(\alpha(g,h)V_{gh}\otimes a_g b_h\right)=\alpha(g,h)\left(U_{gh}\otimes\psi(\rho(V_{gh})\otimes a_g b_h)\right)\\ &= U_gU_h\otimes\psi(\rho(V_gV_h)\otimes a_gb_h)=\left(U_g\otimes\psi(\rho(V_g)\otimes a_g)\right)\left(U_h\otimes\psi(\rho(V_h)\otimes b_h)\right)\\ &= \varphi\left(V_g\otimes a_g\right)\varphi\left(V_h\otimes b_h\right)\end{aligned}$$

φ is a graded non-zero algebra homomorphism and A is G-simple so φ is a graded embedding.

2. Let $\alpha\in Z^2(G,\mathbb{K}^\times)$ and let $H=\left\{h\in G\ \mid\ \frac{\alpha(h,g)}{\alpha(g,h)}=1\ \forall g\in G\right\}$ be the degrees of homogeneous elements in the center of $\mathbb{K}^\alpha G$, then as a non-graded algebra we have ([7] Corollary 13)

$$\mathbb{K}^\alpha G\cong\overbrace{M_d(\mathbb{K})\oplus M_d(\mathbb{K})\oplus\cdots\oplus M_d(\mathbb{K})}^{|H|\ times}\qquad d^2=[G:H]$$

$\mathbb{K}G$ is semisimple abelian algebra, so as a non-graded algebra it is a direct sum of the field $\mathbb{K}$ and we get

$$\begin{aligned}\mathbb{K}^\alpha G\otimes M_{r_1}(\mathbb{K}) &\cong \overbrace{M_{dr_1}(\mathbb{K})\oplus M_{dr_1}(\mathbb{K})\oplus\cdots\oplus M_{dr_1}(\mathbb{K})}^{|H|\ times}\\ \mathbb{K}G\otimes M_{r_2}(\mathbb{K}) &\cong \overbrace{M_{r_2}(\mathbb{K})\oplus M_{r_2}(\mathbb{K})\oplus\cdots\oplus M_{r_2}(\mathbb{K})}^{|G|\ times}\\ (*)\quad Id(M_{r_2}(\mathbb{K}))=Id(\mathbb{K}G\otimes M_{r_2}(\mathbb{K})) &\subseteq Id(\mathbb{K}^\alpha G\otimes M_{r_1}(\mathbb{K}))=Id(M_{dr_1}(\mathbb{K}))\end{aligned}$$

From Amitsur-Levitzki theorem[8] the multilinear identity of $M_n(\mathbb{K})$ with the smallest degree, has degree $2n$, and from the inclusion above we see that $2dr_1\le 2r_2\ \Rightarrow\ dr_1\le r_2$.

□

Since we can assume that $\beta\equiv 1$, then the last lemma proves that (1) $\iff$ (3) and (2) $\Rightarrow$ (3) in 14. (1) $\Rightarrow$ (2) is obvious, and so the theorem is proved for the case where $A=\mathbb{K}^\alpha G\otimes M_{r_1}(\mathbb{K})$.

Before we continue, we note that Lemma 15 part (1) shows that if d is the dimension of the smallest representation of $\mathbb{K}^\alpha G$, then there is a graded embedding $\varphi:\mathbb{K}^\alpha G\to\mathbb{K}G\otimes M_d(\mathbb{K})$. As non-graded algebras $\mathbb{K}^\alpha G$ and $\mathbb{K}G\otimes M_d(\mathbb{K})$ are direct sum of matrix algebras

$$\begin{aligned}\mathbb{K}^\alpha G &\cong M_{d_1}(\mathbb{K})\oplus\cdots\oplus M_{d_n}(\mathbb{K})\\ \mathbb{K}G\otimes M_d(\mathbb{K}) &\cong \left(M_{r_1}(\mathbb{K})\oplus\cdots\oplus M_{r_m}(\mathbb{K})\right)\otimes M_d(\mathbb{K})\cong M_{dr_1}(\mathbb{K})\oplus\cdots\oplus M_{dr_m}\\ & \quad d=d_1\le d_2\le\cdots\le d_n\quad ;\quad 1=r_1\le r_2\le\cdots\le r_m\end{aligned}$$

Let $\pi_j:\mathbb{K}G\otimes M_d(\mathbb{K})\to M_{dr_j}(\mathbb{K})$ be the natural projection. $M_{d_n}(\mathbb{K})$ is not in the kernel of φ (because it is injective), so there is some j such that $M_{d_n}(\mathbb{K})$ is not in the kernel of $\pi_j\circ\varphi$, and since it is simple, then $\pi_j\circ\varphi\mid_{M_{d_n}(\mathbb{K})}$ is injective. From this we get that $d_n\le dr_j\le dr_m\quad\Rightarrow\quad d_n/d\le r_m$. By definition $d=d_1$, and $r_1=1$ because we have the augmentation representation on $\mathbb{K}G$, so we just proved that:

Theorem 16. *Let G be a finite arbitrary group. For $\alpha \in Z^2(G, \mathbb{K}^\times)$ we define $\Phi(\alpha)$ to be the ratio $\frac{d_n}{d_1}$ where d_n is the dimension of the largest representation of $\mathbb{K}^\alpha G$ and d_1 is the dimension of the smallest representation, then Φ achieves its maximum at $\alpha \equiv 1$.*

We return now to the general case. Suppose that $r_2 \geq d\,|\bar{s}|$ as in part (3) in the theorem then $\overline{r_2} \gtrsim_G \bar{d} \times \bar{s}$ since all the elements of $\bar{s}$ are in G. This means that up to this equivalence we can "find" the $M_{\bar{s}}(\mathbb{K})$ part of A in the matrix part of B.

If this idea is true, then the choice of the tuple $\bar{s}$ is irrelevant, and only its size matters. Changing $\bar{s}$ to the tuple $(e, e, ..., e)$ to get the algebra $A' = \mathbb{K}^\alpha H \otimes M_{r_1}(\mathbb{K})$ we reduce the grading of A to the subgroup H. Doing the same with B, we need to reduce the group algebra part from $\mathbb{K}G$ to $\mathbb{K}H$, and this will give a reduction to the first case.

Lemma 17. *Let G be a finite abelian group, $H \leq G$. Let $A = \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$, $B = \mathbb{K}G \otimes M_{r_2}(\mathbb{K})$ where $\bar{s} \in G^{r_1}$ and define $A' = \mathbb{K}^\alpha H \otimes M_{r_1}(\mathbb{K})$, $B' = \mathbb{K}H \otimes M_{r_2}(\mathbb{K}) = B_H$.*

If $Id_G(B) \subseteq Id_G(A)$ then $Id_H(B') \subseteq Id_H(A')$.

Proof. We first recall from Lemma 13 that any multilinear polynomial identity of B' has the form $f(x_{h_1,1}, ..., x_{h_n,n}) = \sum_{\sigma \in S_n} \lambda_\sigma \prod x_{h_{\sigma(i)},\sigma(i)}$ where $\bar{h} = (h_1, ..., h_n) \in H^n$ and $\tilde{f} = \sum \lambda_\sigma \prod x_{\sigma(i)} \in Id(M_{r_2}(\mathbb{K}))$ (we use here that $\beta \equiv 1$). We want to show that f is a graded identity of A'. From multilinearity, $f \in Id_H(\mathbb{K}^\alpha H \otimes M_{r_1}(\mathbb{K}))$ iff for every $a_i, b_i \in [r_1]$, $1 \leq i \leq n$ we have

$$(*) \qquad f\left(V_{h_1} \otimes E_{a_1,b_1}, ..., V_{h_n} \otimes E_{a_n,b_n}\right) = 0$$

Notice that the algebras A and A' are the same and only differ in their grading, so we can think of the assignment above in A instead of A'. Fix now the indices a_i, b_i, then the grading of the assignment above in A is $g_i = \deg_A\left(V_{h_i} \otimes E_{a_i,b_i}\right) = s_{a_i}^{-1} h_i s_{b_i}$. For the same f as above let $\bar{g} = (g_1, ..., g_n) \in G^n$ then using again Lemma 13 and the fact that $\beta \equiv 1$ we get that $f(x_{g_1,1}, ..., x_{g_n,n}) = \sum_{\sigma \in S_n} \lambda_\sigma \prod x_{g_{\sigma(i)},\sigma(i)}$ is an identity of B , because $\tilde{f} \in Id(M_{r_2}(\mathbb{K}))$. This shows that $(*)$ is indeed zero in A, so also in A'. We can do this for any assignment in A' of basis elements (though for each assignment, the tuple $\bar{g}$ may be different). f is multilinear, so $f \in Id_G(A')$ and we are done. □

We are now ready to prove Theorem 14

Proof. The direction (1) $\Rightarrow$ (2) is obvious.

Suppose we have (2) so $Id_G(\mathbb{K}G \otimes M_{r_2}(\mathbb{K})) \subseteq Id_G(\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K}))$. From the last lemma we have $Id_G(\mathbb{K}H \otimes M_{r_2}(\mathbb{K})) \subseteq Id_G(\mathbb{K}^\alpha H \otimes M_{|\bar{s}|}(\mathbb{K}))$ and we can now use Lemma 15 (2) to conclude that $r_2 \geq d\,|s|$. Notice that since $\bar{s} \in G^{r_1}$ then $\bar{d} \times \bar{s} \sim_G \bar{d} \times \overline{r_1}$, so $\overline{r_2} \gtrsim_G \bar{d} \times \bar{s}$ and we proved (2) $\Rightarrow$ (3).

Assume (3). From Lemma 15 (1) we have a graded embedding $\iota : \mathbb{K}^\alpha H \hookrightarrow \mathbb{K}H \otimes M_d(\mathbb{K})$. Since $\overline{r_2} \gtrsim_G \bar{d} \times \bar{s}$, we can assume wlog that $B = \mathbb{K}G \otimes M_d(\mathbb{K}) \otimes M_{\bar{s}}(\mathbb{K})$. Define the function $\varphi : A \to B$ by

$$\varphi(V_h \otimes E_{i,j}) = \iota(V_h) \otimes E_{i,j}$$

This is a graded algebra homomorphism since

$$\begin{aligned}
\deg(\iota(V_h)\otimes E_{i,j}) &= s_i^{-1}\deg(\iota(V_h))s_j = s_i^{-1}hs_j = \deg(V_h\otimes E_{i,j})\\
\varphi(V_{h_1}\otimes E_{i_1,j_1})\varphi(V_{h_2}\otimes E_{i_2,j_2}) &= (\iota(V_{h_1})\otimes E_{i_1,j_1})\,(\iota(V_{h_2})\otimes E_{i_2,j_2}) = \iota(V_{h_1}V_{h_2})\otimes E_{i_1,j_2}\delta_{j_1,i_2}\\
&= \alpha(h_1,h_2)\iota(V_{h_1h_2})\otimes E_{i_1,j_2}\delta_{j_1,i_2} = \alpha(h_1,h_2)\delta_{i_2,j_1}\varphi(V_{h_1h_2}\otimes E_{i_1,j_2})\\
&= \varphi(V_{h_1}V_{h_2}\otimes E_{i_1,j_1}E_{i_2,j_2}) = \varphi((V_{h_1}\otimes E_{i_1,j_1})\,(V_{h_2}\otimes E_{i_2,j_2}))
\end{aligned}$$

$\varphi\neq 0$ and A is simple graded so φ is a graded embedding hence (3) $\Rightarrow$ (1) $\square$

3.2 Part 2 - $A=\mathbb{K}^{\alpha}N_1\otimes M_{\bar{s}}(\mathbb{K})$, $B=\mathbb{K}^{\beta}N_2\otimes M_{\bar{t}}(\mathbb{K})$, $\bar{s}\in N_2^{r_1},\ \bar{t}\in N_1^{r_2}$

Before we go on to the proof of this part we give a notation for better use of the structure of A and B as block matrices. Let $N_1\leq G'\leq G$ and $\bar{s}\in G^{r_1}$ then we can decompose $\bar{s}$ into $\bar{s}=\bar{s}^{(1)}+\bar{s}^{(2)}+\cdots+\bar{s}^{(n)}$ such that $\bar{s}^{(i)}$ is a tuple of elements in $G'g_i$, and $\{g_1,...,g_n\}$ are different right G' coset representatives (in this part we will take $G'=N_1$). With this fixed decomposition, for the algebra $A=\mathbb{K}^{\alpha}N_1\otimes M_{\bar{s}}(\mathbb{K})$ we write

$$\begin{aligned}
M_{g_i,g_j} &= M_{\bar{s}^{(i)},\bar{s}^{(j)}} := span\left\{E_{k,l}\ \mid\ s_k\in G'g_i,\ s_l\in G'g_j\right\}\\
A_{g_i,g_j} &= \mathbb{K}^{\alpha}N_1\otimes M_{g_i,g_j}
\end{aligned}$$

an easy check shows that $M_{g_i,g_j}M_{g_k,g_l}=0$ if $j\neq k$ and $M_{g_i,g_j}M_{g_j,g_l}\subseteq M_{g_i,g_l}$. We also have $\deg(A_{g_i,g_j})\subseteq g_i^{-1}G'g_j=g_i^{-1}g_jG'$ which is a coset of G'.

This gives the decompositions $M_{r_1}(\mathbb{K})=\bigoplus M_{g_i,g_j}$ and $A=\bigoplus A_{g_i,g_j}$ to block matrices. We assume that G is abelian (though it is enough to assume that $G'\trianglelefteq G$) then the blocks on the diagonal have all degrees in $\deg(A_{g_i,g_i})=g_i^{-1}G'g_i=G'$. Suppose that we can find $h\in g_i^{-1}G'g_j\cap G'$ so there is some $\tilde{h}\in G'$ such that

$$h=g_i^{-1}\tilde{h}g_j \quad\Rightarrow\quad hg_i=\tilde{h}g_j$$

so g_i and g_j are in the same right coset of G', hence they are equal. This shows that $A_{G'}$ is exactly the diagonal $\bigoplus_1^n A_{g_i,g_i}$.

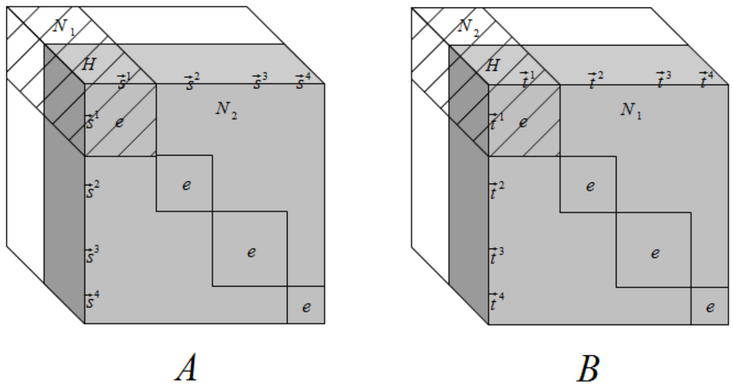
We now check how the rest of the cosets of G' sit inside of A. Notice that

$$\deg(A_{g_i,g_j})\cap\deg(A_{g_i,g_k})\subseteq g_i^{-1}G'g_j\cap g_i^{-1}G'g_k=g_i^{-1}\left(G'g_j\cap G'g_k\right)$$

which is empty unless $g_j=g_k$, so in particular a coset of G' can be in at most one block on the i'th row, and similarly for columns.

Suppose now that $N_1N_2=G$ where $N_1,N_2\leq G$, $A=\mathbb{K}^{\alpha}N_1\otimes M_{\bar{s}}(\mathbb{K})$, $B=\mathbb{K}^{\beta}N_2\otimes M_{\bar{t}}(\mathbb{K})$, $\bar{s}\in N_2^{r_1},\ \bar{t}\in N_1^{r_2}$. We want to see how A_{N_2} (respectively B_{N_1}) sits inside A (respectively B). Suppose that $\deg_A(V_h\otimes E_{i,j})=s_i^{-1}hs_j\in N_2$ for some $h\in N_1,\ s_i,s_j\in N_2$, then $h\in s_iN_2s_j^{-1}=N_2$, and therefore $h\in N_1\cap N_2$. On the other hand, if $h\in N_1\cap N_2$ and $s_i,s_j\in N_2$ then $\deg_A(V_h\otimes E_{i,j})\in N_2$, and we see that $A_{N_2}=\mathbb{K}^{\alpha|_H}H\otimes M_{\bar{s}}(\mathbb{K})$ where $H=N_1\cap N_2$. A similar argument shows that $B_{N_1}=\mathbb{K}^{\beta|_H}H\otimes M_{\bar{t}}(\mathbb{K})$.

To help visualize the structure of A (and B) think of it as a three dimensional cube, where the face in the front is $M_{\bar{s}}(\mathbb{K})$ and as we move on the axis perpendicular to it we change the coefficient in $\mathbb{K}^{\alpha}N_1$, then A (and B) will look like



In the last part, we had only the gray part in A, and only the striped part in B. To extend our embedding we need to find where to send the rest of N_1. If γ is a 2-cocycle on G and the tuple $\bar{g} = (g_1, ..., g_n)$ has all the elements in G exactly once, then there is a natural way to embed $\mathbb{K}^{\gamma}G$ inside $M_{\bar{g}}(\mathbb{K})$ using the right regular representation on $\mathbb{K}^{\gamma}G$. The extension here is done in a similar manner, where we think of ${}^{N_1}/_{H}$ as inside $\mathbb{K}^{\alpha}N_1$ acting by right multiplication on A (with the exact definitions inside the theorem). In this case ${}^{N_1}/_{H}$ will sit inside $M_{\bar{g}}(\mathbb{K})$ where $\bar{g} \in N_1^{[N_1:H]}$ is a tuple of coset representatives of N_1 in H which is also a tuple of coset representatives of N_2 in $G = N_1N_2$. We denote by $\overline{G : N_2}$ a tuple of coset representatives of N_2 in G. Notice that up to equivalence modulo N_2, all such tuples are the same.

Theorem 18. *Let G be a finite abelian group, $N_1, N_2 \leq G$ such that $N_1N_2 = G$, $\alpha \in Z^2(N_1, \mathbb{K}^{\times})$, $\beta \in Z^2(N_2, \mathbb{K}^{\times})$ and $\bar{s} \in N_2^{r_1}$, $\bar{t} \in N_1^{r_2}$. Denote $H = N_1 \cap N_2$, $\alpha' = \alpha \mid_H$, $\beta' = \beta \mid_H$ and let d be the dimension of the smallest representation of $\mathbb{K}^{\alpha'/\beta'}H$. Define $A = \mathbb{K}^{\alpha}N_1 \otimes M_{\bar{s}}(\mathbb{K})$ and $B = \mathbb{K}^{\beta}N_2 \otimes M_{\bar{t}}(\mathbb{K})$ then the following are equal*

1. *There is a graded embedding $A \hookrightarrow B$*
2. *$Id_G(B) \subseteq Id_G(A)$*
3. *$\bar{t} \succsim_{N_2} \bar{d} \times \bar{s} \times \overline{G : N_2}$.*

Proof. The part (1) $\Rightarrow$ (2) is obvious.

Assume now that $Id_G(B) \subseteq Id_G(A)$. Let $T = \{w_1, ..., w_n\} \subseteq N_1$ a set of coset representatives of N_2 in $G = N_1N_2$ (containing the elements of $\bar{t}$) and denote $B_{w_i} = \mathbb{K}^{\beta}N_2 \otimes M_{w_i,w_i} \subseteq B_{N_2}$ and $A' = \mathbb{K}^{\alpha'}H \otimes M_{\bar{s}}(\mathbb{K}) = A_{N_2}$. In the figure above, A' is the gray part, while the B_{w_i} are the block on the diagonal.

The idea that we use here and in the next part is that if B has $n \times n$ blocks and $a_1, ..., a_n$ are diagonal block matrices such that the i'th block of a_i is zero then $\prod a_i = 0$. To use this we will try to find polynomials f_i, such that for each assignment in B, the i'th block will be zero, and since $B_{N_2} = \bigoplus_1^n B_{w_i}$ this is equivalent to $f_i \in Id_{N_2}(B_{w_i})$.

Suppose now that a is a block diagonal matrix with 0 on the i block. Let b, c be two block matrices such that the only non-zero blocks on the $\tilde{i}$'th column (as a block matrix)

of b is in location $(i,\tilde{i})$ and the only non-zero block on the $\tilde{i}$'th row of c is on the $(\tilde{i},i)$ location then

$$(cab)_{\tilde{i},\tilde{i}} = \sum_{j,k} c_{\tilde{i},j} a_{j,k} b_{k,\tilde{i}} = c_{\tilde{i},i} a_{i,i} b_{i,\tilde{i}} = 0$$

Let $g \in G$ be fixed and $b \in B_{gN_2}$. The degrees on the i'th row are $w_i^{-1} N_2 w_j$ where $1 \le j \le n$ so b can have a non-zero block only in location (i,j) where $w_i^{-1} w_j N_2 = gN_2 \iff w_j \in w_i g N_2$. By the remark before the theorem, b's only non-zero block on the j column can be only in the (i,j) location. A similar argument is true for $c \in B_{g^{-1}N_2}$, and if $a \in A_e$ is a diagonal block matrix with zero i block then $cab \in A_e$ is again a block diagonal matrix with a zero j block, so we can move the zero block from the i'th location to the j'th location. This is true for any such $b \in B_{gN_2}$ and $c \in B_{g^{-1}N_2}$ so we can extend this argument to the level of identities.

Back to the proof, we know that $Id_G(B) \subseteq Id_G(A)$, so in particular we have

$$\bigcap Id_{N_2}(B_{w_i}) = Id_{N_2}\left(\bigoplus B_{w_i}\right) = Id_{N_2}(B_{N_2}) \subseteq Id_{N_2}(A_{N_2}) = Id_{N_2}(A')$$

We want to show that not only the intersection of $Id_{N_2}(B_{w_i})$ is in $Id_{N_2}(A')$, but for each i there is an inclusion $Id_{N_2}(B_{w_i}) \subseteq Id_{N_2}(A')$. Suppose by negation that there is some $1 \le i_0 \le n$, $f \in Id_{N_2}(B_{w_{i_0}})$ such that $f \notin Id_{N_2}(A')$. Since $B_{N_2} = \bigoplus_1^n B_{w_i}$ are the block diagonal matrices of B, then for any assignment of B in f the w_{i_0} block will be zero. Let $f_j = x_{w_j^{-1} w_{i_0}} \cdot f \cdot y_{w_{i_0}^{-1} w_j}$ then for any assignment of B in f_j the w_j block on the diagonal will be zero, and so $\tilde{f} = \prod_1^n (f_j \cdot z_{N_2,j})$ will be an identity of B because each element of B_{N_2} is a diagonal block matrix, and each block on the diagonal will be zero.

On the other hand $w_j^{-1} w_{i_0} \in N_1$ for each j and $V_{w_j^{-1} w_{i_0}} \otimes I \in A_{w_j^{-1} w_{i_0}}$ are invertibles so $f_j \notin Id_G(A)$ for every j. For each $1 \le a, b \le |s|$ and $h \in H$ we have $\deg(V_h \otimes E_{a,b}) \in N_2$ so we can find an assignment for $\prod_1^n f_j \cdot z_{N_2,j}$ in A that is not zero, hence $\tilde{f} \notin Id_G(A)$, and we get a contradiction because $Id_G(B) \subseteq Id_G(A)$.

From the contradiction we know that $Id_{N_2}(B_{w_i}) \subseteq Id_{N_2}(A')$ for each i. Write $\bar{t} = \bar{t}^{(1)} + \cdots + \bar{t}^{(n)}$ where the elements of $\bar{t}^{(i)}$ are in $N_2 w_i$. Notice that in order to use the previous part we needed that the elements of $\bar{t}^{(i)}$ are all e, or up to equivalence modulo N_2 all the elements are in N_2. Taking $\bar{t}^{(i)} w_i^{-1}$ will satisfy this condition, and since G is abelian we have $B_{w_i} = \mathbb{K}^\beta N_2 \otimes M_{\bar{t}^{(i)}}(\mathbb{K}) \cong_G \mathbb{K}^\beta N_2 \otimes M_{\bar{t}^{(i)} w_i^{-1}}(\mathbb{K})$. We now use the previous part to show that $\left|\bar{t}^{(i)}\right| = \left|\bar{t}^{(i)} w_i^{-1}\right| \ge |\bar{s}| \cdot d$ for each i therefore

$$\begin{aligned} \bar{t}^{(i)} w_i^{-1} \succsim_{N_2} \bar{s} \times \bar{d} \quad &\Rightarrow \quad \bar{t}^{(i)} \succsim_{N_2} \bar{s} \times \bar{d} \times (w_i) \\ \bar{t} = \bar{t}^{(1)} + \cdots + \bar{t}^{(n)} \succsim_{N_2} \sum_i \bar{s} \times \bar{d} \times (w_i) \quad &= \quad \bar{s} \times \bar{d} \times \sum_i (w_i) = \bar{d} \times \bar{s} \times \overline{G : N_2} \end{aligned}$$

so (2) $\Rightarrow$ (3).

Suppose that condition (3) is true, then wlog we can assume that the tuple $\bar{t}$ is $\bar{d} \times \bar{s} \times \overline{G : N_2}$ or in other words $M_{\bar{t}}(\mathbb{K}) \cong M_d(\mathbb{K}) \otimes M_{\bar{s}}(\mathbb{K}) \otimes M_{\overline{G:N_2}}(\mathbb{K})$.

As we mentioned in the remark before the theorem, we now think of $\mathbb{K}^\alpha N_1 \otimes M_{\bar{s}}(\mathbb{K})$ as a $[G : N_2]$ copies of $\mathbb{K}^{\alpha'} H \otimes M_{\bar{s}}(\mathbb{K})$, and then the right multiplication in $\mathbb{K}^\alpha N_1 \otimes M_{\bar{s}}(\mathbb{K})$ permutes these copies (that are mapped to the diagonal block matrices in B) and also

act on each copy alone.

$$N_1 = N_1 \cap N_1N_2 = N_1 \cap (\uplus N_2 w_i) = \uplus (N_1 \cap N_2 w_i) = \uplus (N_1 \cap N_2) w_i$$

so T is also a set of representatives of $H = N_1 \cap N_2$ in N_1.

We first give a notation. Let $w_i \in T$ then for every $g \in N_1$ there are unique $h \in H$ and $w_j \in T$ such that $w_i g = h w_j$. Denote these h, w_j by $h := h_{w_i,g}$ and $w_j = w_i^g$ (in particular $w_i^e = w_i$ for all i) then

$$\begin{aligned}
w_i(g_1g_2) &= h_{w_i,g_1g_2} w_i^{g_1g_2} \\
(w_i g_1)\, g_2 &= h_{w_i,g_1} w_i^{g_1} g_2 = h_{w_i,g_1} h_{w_i^{g_1},g_2} (w_i^{g_1})^{g_2} \\
&\Rightarrow h_{w_i,g_1} h_{w_i^{g_1},g_2} = h_{i,g_1g_2} \qquad w_i^{g_1g_2} = (w_i^{g_1})^{g_2}
\end{aligned}$$

and in particular $G \to Aut(T)$ defined above is a homomorphism. Notice that for N_1 abelian and $g \in H$ we get that $w_i g = g w_i$ so $w_i^g = w_i$ and $h_{w_i,g} = g$ for any $w_i \in T$.

We would like to view N_1 as a group that acts on itself where "itself" is $\sqcup H w_i$. Each element in N_1 can be written uniquely as $h w_i$ where $w_i \in T$ and $h \in H$. To make calculations easier, we choose a new basis for $\mathbb{K}^\alpha N_1$, such that V_{w_i} can be any choice, except $V_e = 1$, and V_h can be any choice for $h \in H \setminus \{e\}$ and then for each $h \in H$ and $w_i \in T$ we let $V_h V_{w_i} = V_{h w_i}$, or in other words $\alpha(h, w_i) = 1$ for every $h \in H$ and $w_i \in T$.

For $w_i \in T,\ h \in H$, $g \in N_1$ and $b, c \in M_{\bar{s}}(\mathbb{K})$ we have

$$(V_h V_{w_i} \otimes b)(V_g \otimes c) = \alpha(w_i, g) V_h V_{w_i g} \otimes bc = \alpha(w_i, g) V_h V_{h_{w_i,g} w_i^g} \otimes bc = \alpha(w_i, g) \left(V_h V_{h_{w_i,g}} \otimes bc\right) \left(V_{w_i^g} \otimes 1\right)$$

From the previous part, we have a graded embedding $\varphi : \mathbb{K}^{\alpha'} H \otimes M_{\bar{s}}(\mathbb{K}) \to \mathbb{K}^\beta N_2 \otimes M_{|s|d}(\mathbb{K})$. We now define a mapping

$$\begin{aligned}
\phi : \mathbb{K}^\alpha N_1 \otimes M_{\bar{s}}(\mathbb{K}) &\to \mathbb{K}^\beta N_2 \otimes M_{\bar{t}}(\mathbb{K}) \cong \mathbb{K}^\beta N_2 \otimes M_{d|s|}(\mathbb{K}) \otimes M_{\overline{G:N_2}}(\mathbb{K}) \\
\phi(V_g \otimes c) &= \sum_{w_i \in T} \alpha(w_i, g) \varphi\left(V_{h_{w_i,g}} \otimes c\right) \otimes E_{w_i, w_i^g}
\end{aligned}$$

where we identify in the $M_{\overline{G:N_2}}(\mathbb{K})$ part the set of indices $\{1, ..., [G : N_2]\}$ with the set T of coset representatives. φ is graded so the degree of each summand is

$$w_i^{-1} h_{w_i,g} \deg_A(c) w_i^g = w_i^{-1} w_i g \cdot \deg_A(c) = g \cdot \deg_A(c) = \deg_A(V_g \otimes c)$$

hence ϕ is graded. The linear extension of this map is a vector space homomorphism and we want it to be also multiplicative. We write V_g for $V_g \otimes I$ and then

$$\begin{aligned}
\phi(V_{g_1})\phi(V_{g_2}) &= \left(\sum_{w_i \in T} \alpha(w_i, g_1) \varphi\left(V_{h_{w_i,g_1}}\right) \otimes E_{w_i, w_i^{g_1}}\right) \left(\sum_{w_j \in T} \alpha(w_j, g_2) \varphi\left(V_{h_{w_j,g_2}}\right) \otimes E_{w_j, w_j^{g_2}}\right) \\
&= \sum_{w_i \in T} \alpha(w_i, g_1) \alpha(w_i^{g_1}, g_2) \varphi\left(V_{h_{w_i,g_1}}\right) \varphi\left(V_{h_{w_i^{g_1},g_2}}\right) \otimes E_{w_i, (w_i^{g_1})^{g_2}} \\
&= \sum_{w_i \in T} \alpha(w_i, g_1) \alpha(w_i^{g_1}, g_2) \alpha(h_{w_i,g_1}, h_{w_i^{g_1},g_2}) \varphi\left(V_{h_{w_i,g_1g_2}}\right) \otimes E_{w_i, w_i^{g_1g_2}} = (*)
\end{aligned}$$

We now use the 2-cocycle property of α together with $\alpha(h,w_i)=1$ for all $h\in H$ and $w_i\in T$ to get

$$\begin{aligned}
\alpha(hw_j,g) &= \alpha(hw_j,g)\alpha(h,w_j)=\alpha(h,w_jg)\alpha(w_j,g)=\alpha(h,h_{w_j,g}w_j^g)\alpha(w_j,g)\\
\alpha(h,h_{w_j,g}w_j^g) &= \alpha(h,h_{w_j,g}w_j^g)\alpha(h_{w_j,g},w_j^g)=\alpha(h,h_{w_j,g})\alpha(hh_{w_j,g},w_j^g)=\alpha(h,h_{w_j,g})\\
\Rightarrow \alpha(h,h_{w_j,g})\alpha(w_j,g) &= \alpha(hw_j,g)\\
\Rightarrow \alpha(h_{w_i,g_1},h_{w_i^{g_1},g_2})\alpha(w_i^{g_1},g_2) &= \alpha(h_{w_i,g_1}w_i^{g_1},g_2)=\alpha(w_ig_1,g_2)
\end{aligned}$$

$$\begin{aligned}
(*) &= \sum_{w_i\in T}\alpha(w_i,g_1)\alpha(w_ig_1,g_2)\varphi\left(V_{h_{w_i,g_1g_2}}\right)\otimes E_{w_i,w_i^{g_1g_2}}=\alpha(g_1,g_2)\sum_{w_i\in T}\alpha(w_i,g_1g_2)\varphi\left(V_{h_{w_i,g_1g_2}}\right)\otimes E_{w_i,w_i^{g_1g_2}}\\
&= \alpha(g_1,g_2)\phi(V_{g_1g_2})=\phi(V_{g_1}V_{g_2})
\end{aligned}$$

From the definition of ϕ we get that $\phi(V_g\otimes c)=\phi(V_g)\left(\varphi(1\otimes c)\otimes I\right)=\left(\varphi(1\otimes c)\otimes I\right)\phi(V_g)$ (using the multiplicativity of φ) and then

$$\begin{aligned}
\phi(V_{g_1}\otimes c_1)\phi(V_{g_2}\otimes c_2) &= \phi(V_{g_1})\varphi(1\otimes c_1)\phi(V_{g_2})\varphi(1\otimes c_2)=\phi(V_{g_1})\phi(V_{g_2})\varphi(1\otimes c_1)\varphi(1\otimes c_2)\\
&= \phi(V_{g_1}V_{g_2})\varphi(1\otimes c_1c_2)=\phi(V_{g_1}V_{g_2}\otimes c_1c_2)
\end{aligned}$$

so ϕ is multiplicative.

ϕ is a non-zero graded algebra homomorphism from A to B and A is G-simple so ϕ is a graded embedding, and this proves part (3) $\Rightarrow$ (1) and completes the theorem. Notice that if we view B as

$$B\cong\mathbb{K}^{\beta}N_2\otimes M_d(\mathbb{K})\otimes M_{|\bar{s}|}(\mathbb{K})\otimes M_{\overline{G:N_2}}(\mathbb{K})\cong\mathbb{K}^{\beta}N_2\otimes M_d(\mathbb{K})\otimes M_{\bar{s}}(\mathbb{K})\otimes M_{\overline{G:N_2}}(\mathbb{K})$$

and $\iota:\mathbb{K}^{\alpha'}H\to\mathbb{K}^{\beta'}H\otimes M_d(\mathbb{K})$ the graded embedding from step 1 then

$$\begin{aligned}
\phi(V_g\otimes c) &= \sum_{w_i\in T}\alpha(w_i,g)\varphi\left(V_{h_{w_i,g}}\otimes c\right)\otimes E_{w_i,w_i^g}=\sum_{w_i\in T}\alpha(w_i,g)\iota(V_{h_{w_i,g}})\otimes c\otimes E_{w_i,w_i^g}\\
\phi(V_e\otimes c) &= \sum_{w_i\in T}\alpha(w_i,e)\varphi\left(V_{h_{w_i,e}}\otimes c\right)\otimes E_{w_i,w_i^e}=\sum_{w_i\in T}\varphi\left(V_e\otimes c\right)\otimes E_{w_i,w_i}=\iota(V_e)\otimes c\otimes I
\end{aligned}$$

□

3.3 Part 3 - $A=\mathbb{K}^{\alpha}N_1\otimes M_{\bar{s}}(\mathbb{K})$, $B=\mathbb{K}^{\beta}N_2\otimes M_{\bar{t}}(\mathbb{K})$

As we did in the previous part, we are going to think of A and B as block matrices, such that $G'=N_1N_2$ will be the blocks on the diagonal. We will then try to match each such block of degree G' in A to some block with degree G' in B and then "extend" the embedding to all of A.

Theorem 19. *Let G be an abelian group, $N_1,N_2\le G$ be finite subgroups of G , $\alpha\in Z^2(N_1,\mathbb{K}^{\times})$, $\beta\in Z^2(N_2,\mathbb{K}^{\times})$ and $\bar{s}\in G^{r_1}$, $\bar{t}\in G^{r_2}$. Let $H=N_1\cap N_2$, $\alpha'=\alpha\mid_H$, $\beta'=\beta\mid_H$ and let d be the dimension of the smallest representation of $\mathbb{K}^{\alpha'/\beta'}H$, then the following conditions are equal for $A=\mathbb{K}^{\alpha}N_1\otimes M_{\bar{s}}(\mathbb{K})$ and $B=\mathbb{K}^{\beta}N_2\otimes M_{\bar{t}}(\mathbb{K})$*

1. *There is a graded embedding $A\hookrightarrow B$*

2. $Id_G(B) \subseteq Id_G(A)$

3. $\exists g \in G, \quad g\bar{t} \gtrsim_{N_2} d \times \overline{G' : N_2} \times \tilde{s}$ *where* $\bar{s} \sim_{N_1} \tilde{s}$.

Proof. The part (1) $\Rightarrow$ (2) is obvious.

Suppose that (3) is true so we can assume wlog that $\bar{t} = d \times \overline{G' : N_2} \times \bar{s}$ and therefore $B = \mathbb{K}^{\beta} N_2 \otimes M_d(\mathbb{K}) \otimes M_{\overline{G':N_2}}(\mathbb{K}) \otimes M_{\bar{s}}(\mathbb{K})$. From the previous part, we can find a graded embedding $\phi_1 : \mathbb{K}^{\alpha} N_1 \to \mathbb{K}^{\beta} N_2 \otimes M_d(\mathbb{K}) \otimes M_{\overline{N_1 N_2 : N_2}}(\mathbb{K})$. Define the map $\phi : A \to B$ by

$$\phi(V_g \otimes E_{i,j}) = \phi_1(V_g) \otimes E_{i,j}$$

and extend linearly. This is a graded homomorphism since

$$\deg_B (\phi_1(V_g) \otimes E_{i,j}) \quad = \quad s_i^{-1} \deg(\phi_1(V_g))\, s_j = s_i^{-1} g s_j = \deg_A (V_g \otimes E_{i,j})$$

where we used the fact that ϕ_1 is graded.

$$\begin{aligned} \phi(V_{g_1} \otimes E_{i_1,j_1}) \phi(V_{g_2} \otimes E_{i_2,j_2}) &= (\phi_1(V_{g_1}) \otimes E_{i_1,j_1}) \cdot (\phi_1(V_{g_2}) \otimes E_{i_2,j_2}) = \delta_{j_1,i_2} (\phi_1(V_{g_1}) \phi_1(V_{g_2}) \otimes E_{i_1,j_2}) \\ &= \delta_{j_1,i_2} \alpha(g_1, g_2) (\phi_1(V_{g_1 g_2}) \otimes E_{i_1,j_2}) = \delta_{j_1,i_2} \alpha(g_1, g_2) \phi (V_{g_1 g_2} \otimes E_{i_1,j_2}) \\ &= \phi ((V_{g_1} \otimes E_{i_1,j_1})(V_{g_2} \otimes E_{i_2,j_2})) \end{aligned}$$

ϕ is a non-zero graded algebra homomorphism and A is G-simple so it is a graded embedding.

Suppose now that condition (2) is true - $Id_G(B) \subseteq Id_G(A)$. Decompose $\bar{s}$ to $\bar{s} = \left(\bar{s}^{(1)}, ..., \bar{s}^{(n)}\right)$ such that the elements of $\bar{s}^{(i)}$ (non-empty tuple) are in $G' u_i$ and $U = \{u_1, ..., u_n\}$ are different coset representatives of G' in G, and define $\bar{t} = \left(\bar{t}^{(1)}, ..., \bar{t}^{(m)}\right)$ and $V = \{v_1, ..., v_m\}$ similarly. We write

$$\begin{aligned} A_i &:= \mathbb{K}^{\alpha} N_1 \otimes M_{\bar{s}^{(i)}, \bar{s}^{(i)}} \cong_G \mathbb{K}^{\alpha} N_1 \otimes M_{\bar{s}^{(i)} u_i^{-1}, \bar{s}^{(i)} u_i^{-1}} \\ B_i &:= \mathbb{K}^{\beta} N_2 \otimes M_{\bar{t}^{(i)}, \bar{t}^{(i)}} \cong_G \mathbb{K}^{\beta} N_2 \otimes M_{\bar{t}^{(i)} v_i^{-1}, \bar{t}^{(i)} v_i^{-1}} \end{aligned}$$

and we want to show that $m \geq n$ and up to an injective function $\sigma : [m] \to [n]$ we have $Id_{G'}(B_{\sigma(i)}) \subseteq Id_{G'}(A_i)$. Notice that $A_{G'} = \bigoplus A_i$ and $B_{G'} = \bigoplus B_j$.

For each $1 \leq i \leq n$ and $1 \leq j \leq m$ if $Id_{G'}(B_j) \not\subseteq Id_{G'}(A_i)$ then choose some polynomial $f_{i,j} \in Id_{G'}(B_j) \backslash Id_{G'}(A_i)$. For i fixed define $f_i = x_{G',0} \cdot \prod_j [f_{i,j} \cdot x_{G',j}]$, then for $1 \leq j_0 \leq m$ if $Id_{G'}(B_{j_0}) \not\subseteq Id_{G'}(A_i)$ then f_{i,j_0} appears in this product, and therefore, for every assignment of B in f_i, the j_0 block on the diagonal will be zero. In other words, if there is an assignment of f_i such that the j_0 block is not zero then $Id_{G'}(B_j) \subseteq Id_{G'}(A_i)$.

Define

$$f = \prod_{i=1}^{n} y_{u_1^{-1} u_i G', i} \cdot f_i \cdot z_{u_i^{-1} u_1 G', i}$$

For each i, j, the polynomial $f_{i,j} \notin Id_{G'}(A_i)$ and A_i is simple so $f_i \notin Id_{G'}(A_i)$ (Lemma 24) and therefore there is an assignment $\bar{a}_i$ for f_i such that the u_i block on the diagonal is not zero. An assignment in $y_{u_1^{-1} u_i G', i} \cdot f_i \cdot z_{u_i^{-1} u_1 G', i}$ from A will move the u_i block on the diagonal to the u_1 block, and since

$$\begin{aligned} \deg (V_e \otimes M_{u_1, u_i}) &= u_1 u_i^{-1}, \qquad \deg \left(V_e \otimes M_{u_j, u_1}\right) = u_i^{-1} u_1 \\ M_{u_1, u_i} &= M_{u_1, u_1} M_{u_1, u_i} \end{aligned}$$

then $M_{u_1,u_i}f(\bar{a}_i)M_{u_i,u_1}$ has a non-zero u_1 block and then

$$\prod_1^n \left[M_{u_1,u_1}\left(M_{u_1,u_i}f(\bar{a}_i)M_{u_i,u_1}\right)\right]$$

still has a non-zero u_1 block so $f \notin Id_G(A)$ and therefore $f \notin Id_G(B)$.

A general assignment of B in $y_{u_1^{-1}u_iG',i} \cdot f_i \cdot z_{u_i^{-1}u_1G',i}$ moves the v_j block after the assignment in f_i to v_{j_0} block where $v_{j_0}^{-1}v_jG' = u_1^{-1}u_iG'$ if there is such a j_0, and other wise this block goes to zero. $f \notin Id_G(B)$ so there is a block v_{j_0} which is not zero for some assignment in f. Let $\sigma(i)$ be the index such that $v_{\sigma(i)}G' = u_1^{-1}u_iv_{j_0}G'$, then from the remark above the $\sigma(i)$ block is not zero in the assignment of f_i, so $Id_{G'}(B_{\sigma(i)}) \subseteq Id_{G'}(A_i)$. Notice that

$$\sigma(i_1) = \sigma(i_2) \;\Rightarrow\; u_1^{-1}u_{i_1}v_{j_0}G' = u_1^{-1}u_{i_2}v_{j_0}G' \;\Rightarrow\; u_{i_1}G' = u_{i_2}G' \;\Rightarrow\; i_1 = i_2$$

because the u_i are different coset representatives, so σ is injective and wlog we can assume that $\sigma(i) = i$. We now move to a graded sub algebra of B by looking only on the sub tuple $\bar{t} = (\bar{t}^{(1)}, ..., \bar{t}^{(n)})$. Let $g \in G$ such that $v_1g = u_1$ then

$$\begin{aligned} v_1G' &= u_1^{-1}u_1v_{j_0}G' \;\Rightarrow\; v_1G' = v_{j_0}G' \;\Rightarrow\; v_{j_0} = v_1 \\ v_iG' &= u_1^{-1}u_iv_1G' = u_ig^{-1}G' \end{aligned}$$

so by multiplying $\bar{t}$ by g we see that the elements of $g\bar{t}^{(i)}$ and $\bar{s}^{(i)}$ are in in the same coset u_iG'.

$g\bar{t}^{(i)}u_i^{-1}$ and $\bar{s}^{(i)}u_i^{-1}$ are in $G' = N_1N_2$ and so we can find $g\bar{t}^{(i)}u_i^{-1} \sim_{N_2} \tilde{t}^{(i)}$ and $\bar{s}^{(i)}u_i^{-1} \sim_{N_1} \tilde{s}^{(i)}$ such that the elements of $\tilde{t}^{(i)}$ are in N_1 and the elements of $\tilde{s}^{(i)}$ are in N_2. From the previous part, since $Id_{G'}(B_i) \subseteq Id_{G'}(A_i)$ we get

$$g\bar{t}^{(i)}u_i^{-1} \sim_{N_2} \tilde{t}^{(i)} \succsim_{N_2} \bar{d} \times \tilde{s}^{(i)} \times \overline{G' : N_2}$$

$$\begin{aligned} g\bar{t} &= g\bar{t}^{(1)} + \cdots + g\bar{t}^{(n)} = g\bar{t}^{(1)}u_1^{-1}u_1 + \cdots + g\bar{t}^{(n)}u_n^{-1}u_n \\ &\succsim_{N_2} \left(\bar{d} \times \tilde{s}^{(1)} \times \overline{G' : N_2}\right)u_1 + \cdots + \left(\bar{d} \times \tilde{s}^{(n)} \times \overline{G' : N_2}\right)u_n = \bar{d} \times (\tilde{s}^{(1)}u_1 + \cdots \tilde{s}^{(n)}u_n) \times \overline{G' : N_2} \\ \bar{s} &= (\bar{s}^{(1)} + \cdots \bar{s}^{(n)}) = (\bar{s}^{(1)}u_1^{-1}u_1 + \cdots \bar{s}^{(n)}u_n^{-1}u_n) \sim_{N_1} \left(\tilde{s}^{(1)}u_1 + \cdots + \tilde{s}^{(n)}u_n\right) \end{aligned}$$

so (2) $\Rightarrow$ (3), and the theorem is proved. □

By the structure theorem of finite dimensional G-simple algebras, all such algebras are of the form $\mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$ where $H \leq G$ and $\bar{s} \in G^r$ for some r so we have:

Theorem 20. *Let A, B be two finite dimensional G-simple algebra, where G is an abelian group, then there is a graded embedding $A \hookrightarrow B$ iff $Id_G(B) \subseteq Id_G(A)$.*

4 The Non-Abelian Case

We concentrate now on a special case of Theorem 19 - the one where $N_2 = \{e\}$. The theorem shows that if $A = \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$ and $B = M_{\bar{t}}(\mathbb{K})$ where G is abelian, then there is a graded embedding $A \hookrightarrow B$ iff $Id_G(B) \subseteq Id_G(A)$ iff there exists $g \in G$ such that $g \cdot \bar{t} \succsim \bar{H} \times \tilde{s}$, $\tilde{s} \sim_H \bar{s}$ where $\bar{H}$ is a tuple such that each element in H appears in $\bar{H}$ exactly once. We now show that in this case we can omit the condition of commutativity.

Theorem 21. *Let $A = \mathbb{K}^\alpha H \otimes M_{\bar{s}}(\mathbb{K})$, $B = M_{\bar{t}}(\mathbb{K})$ be G-graded algebras, G arbitrary, then the following are equal*

1. *There is a graded embedding $A \hookrightarrow B$*
2. *$Id_G(B) \subseteq Id_G(A)$*
3. *$\exists g \in G$, s.t. $g \cdot \bar{t} \succsim \overline{H} \times \tilde{s}$ and $\tilde{s} \sim_H \bar{s}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

Suppose we have $Id_G(B) \subseteq Id_G(A)$. Write $\tilde{s} \sim_H \bar{s} = (s_1 \times r_1, ..., s_n \times r_n)$ where $\{s_i\}_1^n$ are distinct right coset representatives of H. For each $1 \leq i \leq n$ and each $h \in H$ we define

$$f_{i,h} = x_{s_1^{-1} h s_i} \cdot St_{2r_i - 1}(z_{e,1}, ..., z_{e,2r_i-1}) \cdot y_{(s_1^{-1} h s_i)^{-1}}$$

From the Amitsur-Levitzki theorem, there is an assignment for the $z_{e,j}$ such that the s_i block on the diagonal of A is not zero in $St_{2r_i-1}(z_{e,1}, ..., z_{e,2r_i-1})$, and by choosing suitable assignment for $x_{s_1^{-1} h s_i}$ and $y_{(s_1^{-1} h s_i)^{-1}}$ we see that the s_1 block on the diagonal is not zero in $f_{i,h}$. Define

$$f = \prod_{i=1}^{n} \prod_{h \in H} f_{i,h} \cdot x_{e,(i,h)}$$

where each $f_{i,h}$ is considered with different indeterminates. Since there is an assignment for each $f_{i,h}$ where the s_1 block is not zero, and it is isomorphic to $M_{r_1}(\mathbb{K})$ which is simple, we get that f is not an identity of A.

$Id_G(B) \subseteq Id_G(A)$ so $f \notin Id_G(B)$. f has degree e, so if $\bar{t} \sim (t_1 \times k_1, ..., t_m \times k_m)$ where the $\{t_i\}_1^m$ are distinct, then there is an assignment $\bar{b}$ for f and some j such that the t_j block is not zero in $f(\bar{b})$ and by permuting the elements of $\bar{t}$ we can assume that $j = 1$. The t_1 block on the diagonal of $f(\bar{b})$ is not zero, so it cannot be zero in $f_{i,h}(\bar{b})$ for each i, h, and this can be true only if the $t_{\sigma(i,h)}$ block on the diagonal of $St_{2r_i-1}(\bar{b})$ is not zero where $t_1^{-1} t_{\sigma(i,h)} = s_1^{-1} h s_i$. By the Amitsur-Levitzki theorem we get that $k_{\sigma(i,h)} \geq r_i$. Notice that $\sigma(i,h)$ is injective since

$$\sigma(i,h) = \sigma(i',h') \;\Rightarrow\; s_1^{-1} h s_i = s_1^{-1} h' s_{i'} \;\Rightarrow\; h s_i = h' s_{i'}$$

but the s_i are distinct H right coset representatives so $s_i = s_{i'} \iff i = i'$ and $h = h'$. We therefore have

$$\bar{t} \succsim \sum_i \sum_{h \in H} \left(t_{\sigma(i,h)} \times k_{\sigma(i,h)}\right) \succsim \sum_i \sum_{h \in H} \left(t_1 s_1^{-1} h s_i \times r_i\right) = t_1 s_1^{-1} \sum_i \sum_{h \in H} (h s_i \times r_i) = t_1 s_1^{-1} \cdot \left(\bar{H} \times \bar{s}\right)$$

and the required g for (3) is $s_1 t_1^{-1}$.

Suppose that (3) is true, so wlog we can assume that $\bar{t} = \bar{H} \times \bar{s}$. Let $\varphi : \mathbb{K}^{\alpha} H \to M_{\overline{H}}(\mathbb{K})$ be the right regular representation defined by

$$U_h \mapsto \sum_{h_i \in H} \alpha(h_i, h) E_{h_i, h_i h}$$

where we identify $\{1, ..., |H|\}$ with H. Define the algebra homomorphism $\psi : A \to B$ by $\psi(U_h \otimes E_{k,l}) = \varphi(U_h) \otimes E_{k,l}$.

$$\deg\left(\psi(U_h \otimes E_{k,l})\right) = \deg\left(\varphi(U_h) \otimes E_{k,l}\right) = s_k^{-1} \deg\left(\varphi(U_h)\right) s_l = s_k^{-1} h s_l = \deg(U_h \otimes E_{k,l})$$

so we get that ψ is a graded homomorphism, and from the simplicity of A, it is a graded embedding. $\square$

We can now use again the structure theorem for finite G-simple algebras to conclude

Corollary 22. *Let G be an arbitrary group, $B = M_{\bar{s}}(\mathbb{K})$ where $\bar{s} \in G^r$ and A a finite dimensional G-simple algebra then there is a graded embedding $A \hookrightarrow B$ iff $Id_G(B) \subseteq Id_G(A)$.*

5 Graded Embeddings of a Semisimple Algebra in a Finite Dimensional Algebra

In the previous section we showed that if A, B are finite dimensional G-simple algebras for G abelian then there is a graded embedding $A \hookrightarrow B$ iff $Id_G(B) \subseteq Id_G(A)$.

We would first like to show that we cannot remove the simplicity of A. One easy counter example is this - for any finite dimensional G-graded algebra B (simple or not), let $A = B \oplus B$ with the grading $A_g = B_g \oplus B_g$ then $Id_G(A) = Id_G(B) \cap Id_G(B) = Id_G(B)$ and in particular we have $Id_G(B) \subseteq Id_G(A)$. On the other hand there is no graded embedding $A \hookrightarrow B$ from the simple fact that $\dim(A) = 2\dim(B) > \dim(B)$.

One might think that the theorem fails because A contains two copies of the same object, but even if A is a direct sum of two non-isomorphic simple algebras the theorem fails

Example 23. Let $B = M_{n+2}(\mathbb{K})$ be graded elementary by the group $G = \mathbb{Z}/10\mathbb{Z}$ with the tuple $(0, 1, 1, ..., 1, 3)$, then B is G-simple. Let A_1, A_2 be two sub algebras of B correspond to the sub tuple $(0, 1, 1, ..., 1)$ and the sub tuple $(1, 1, ..., 1, 3)$ then we have $Id_G(B) \subseteq Id_G(A_i)$ for $i = 1, 2$. A_i are also G-simple and

$$Id_G(B) \subseteq Id_G(A_1) \cap Id_G(A_2) = Id_G(A_1 \oplus A_2)$$

Let $A = A_1 \oplus A_2$ and notice that $Id_G(A_1) \neq Id_G(A_2)$, for example because $Supp_G(A_1) = \{0, 1, 9\}$ while $Supp_G(A_2) = \{0, 2, 8\}$.

$$\dim(A_1 \oplus A_2) = \dim(A_1) + \dim(A_2) = 2(1+n)^2 > (2+n)^2 = \dim(A)$$

for large n, so it is not possible that $A = A_1 \oplus A_2 \hookrightarrow B$.

Notice that though in the last example we cannot embed A, which is semisimple, in B, we can embed it in $B \oplus B$. One of the results of this section is that this is always the case.

We now remark that unless otherwise mentioned, there is no condition on the field $\mathbb{K}$ or the group G.

We first give an important property that simple algebras enjoy.

Suppose that $a, b \in M_n(\mathbb{K})$ then we can write

$$a = \sum a_{i,j} E_{i,j} \qquad b = \sum b_{i,j} E_{i,j} \qquad a_{i,j}, b_{i,j} \in \mathbb{K}$$

If a, b are not zero, then there are $i_1, j_1, i_2, j_2 \in \{1, ..., n\}$ such that $a_{i_1,j_1} \neq 0$ and $b_{i_2,j_2} \neq 0$ so

$$E_{1,i_1} \cdot a \cdot E_{j_1,i_2} \cdot b \cdot E_{j_2,1} = a_{i_1,j_1} \cdot b_{i_2,j_2} \cdot E_{1,1} \neq 0 \quad \Rightarrow \quad aE_{j_1,i_2}b \neq 0$$

This condition, that if $a, b \neq 0$ then we can find x such that $axb \neq 0$ is true for any simple graded algebra and we state it here.

Lemma 24. *Let G be any group and $\mathbb{K}$ any field. Let A be a G-simple $\mathbb{K}$ algebra*

1. *Suppose that $a, b \in A$ are non-zero homogeneous elements, then we can find $x \in A$ homogeneous such that $axb \neq 0$.*

2. *Let $f_1, f_2 \notin Id_G(A)$ homogeneous polynomials then there is a homogeneous polynomial h such that $f_1 \cdot h \cdot f_2 \notin Id_G(A)$.*

Proof. 1. Notice that $I = \{a \in A \mid aA = 0\}$ is a graded ideal in A. Since $A^2 \neq 0$ then $I \neq A$, and from simplicity $I = 0$, or in other words, if $a \neq 0$ then $aA \neq 0$. For a nonzero homogeneous element $a \in A$, the set $J = \{b \in A \mid aAb = 0\}$ is again a graded ideal in A. Find $0 \neq a' \in aA$, so $a'A \neq 0$, and therefore $J \neq A$, and from simplicity $J = 0$. If $a, b \neq 0$ are homogeneous then $aAb \neq 0$, and in particular we can find homogeneous $x \in A$ such that $axb \neq 0$.

2. Obvious from the first part of the lemma.

□

The algebras in this section are direct sums of simple algebra, so we would like to see what it means to have an ideal of one such algebra inside another. We recall again that $Id_G(\bigoplus_1^n B_i) = \bigcap_1^n Id_G(B_i)$ for any algebras B_i. In general, if $\bigcap_1^n Id_G(B_i) \subseteq Id_G(A)$, then it doesn't mean that there is some i such that $Id_G(B_i) \subseteq Id_G(A)$. The next lemma shows that this is true, if we also assume that A is simple.

Lemma 25. *Suppose that $B_1, ..., B_n$ are G-graded algebras and A is a G-simple algebra such that $Id_G(\bigoplus B_i) = \bigcap Id_G(B_i) \subseteq Id_G(A)$, then there is $i \in [n]$ such that $Id_G(B_i) \subseteq Id_G(A)$.*

Proof. Suppose that the claim is false, then for each i there is a homogeneous polynomial $f_i \in Id_G(B_i) \backslash Id_G(A)$, but then from Lemma 24 we could find $h_i \in \mathbb{F}\langle X_G \rangle$ such that $f = \prod (f_i \cdot h_i) \notin Id_G(A)$. On the other hand $f \in Id_G(B_i)$ for each i so $f \in \bigcap Id_G(B_i) \subseteq Id_G(A)$ and we get a contradiction. □

We now have a semisimple algebra $A = A_1 \oplus \cdots \oplus A_n$ with $Id_G(A) = \bigcap Id_G(A_i)$ so we might remove some of the algebras A_i without increasing the ideal of identities. If A_1, A_2 are G-graded and $Id_G(A_1) \subseteq Id_G(A_2)$, then the ideal of identities cannot separate between $A_1 \oplus A_2$ and A_1 because $Id_G(A_1 \oplus A_2) = Id_G(A_1) \cap Id_G(A_2) = Id_G(A_1)$. To avoid such cases we give the next definition

Definition 26. Let $A_1, ..., A_n$ be distinct G-graded algebras. We call the set $\{A_i \mid 1 \leq i \leq n\}$ minimal with respect to identities (or just minimal) if for each $1 \leq j \leq n$ we have $\bigcap_{k=1}^{n} Id_G(A_k) \subsetneq \bigcap_{k \neq j} Id_G(A_k)$.

If there are A_i, A_j in the set S of graded algebras such that $Id_G(A_i) \subseteq Id_G(A_j)$ then S is not minimal. The other direction is not true in general, but if the algebras are simple then the last lemma shows that it is true. If $S = \{A_1, ..., A_n\}$ is a set of G-simple algebras that is not minimal, then wlog $\bigcap_2^n Id_G(A_i) = \bigcap_1^n Id_G(A_j)$ so $\bigcap_2^n Id_G(A_i) \subseteq Id_G(A_1)$, but then Lemma 25 shows that there is $i \in \{2, ..., n\}$ such that $Id_G(A_i) \subseteq Id_G(A_1)$. In other words, the set S is minimal iff the ideals $Id_G(A_i)$ are all distinct and there are no inclusions $Id_G(A_i) \subseteq Id_G(A_j)$ for $i \neq j$.

We can now take a condition such as $Id_G(\bigoplus B_j) \subseteq Id_G(\bigoplus A_i)$ and decompose it into "smaller" conditions about each A_i separately.

Lemma 27. *Let $A_1, ..., A_n$ and $B_1, ..., B_m$ be G-simple algebras and define $A = \bigoplus_1^n A_i$, $B = \bigoplus_1^m B_j$*

1. *If $Id_G(B) \subseteq Id_G(A)$ then there is a function $\tau : [n] \to [m]$ such that $Id_G(B_{\tau(i)}) \subseteq Id_G(A_i)$.*

2. *If $Id_G(A) = Id_G(B)$ and the sets $\{A_i\}_1^n$ and $\{B_j\}_1^m$ are minimal, then $m = n$ and there is a permutation $\tau : [n] \to [n]$ such that $Id_G(A_i) = Id_G(B_{\tau(i)})$*

Proof. 1. Let $i \in [n]$ be fixed then

$$\bigcap_1^m Id_G(B_j) = Id_G(B) \subseteq Id_G(A) = \bigcap_1^n Id_G(A_k) \subseteq Id(A_i)$$

We can now use 25 to find some $j \in [m]$ such that $Id_G(B_j) \subseteq Id(A_i)$ - denote this j by $j = \tau(i)$.

2. From part (1) we can find functions $\tau : [n] \to [m]$ and $\pi : [m] \to [n]$ such that $Id_G(B_{\tau(i)}) \subseteq Id_G(A_i)$ and $Id_G(A_{\pi(j)}) \subseteq Id_G(B_j)$, so for all i

$$Id_G(A_{\pi(\tau(i))}) \subseteq Id_G(B_{\tau(i)}) \subseteq Id_G(A_i)$$

but the set $\{A_i\}_1^n$ is minimal and the A_i are simple, so from the remark before the lemma we must have $\pi(\tau(i)) = i$ and in particular $Id_G(B_{\tau(i)}) = Id_G(A_i)$, and τ is injective. The same is true in the other direction, so π is injective and therefore $n = m$, and τ is a permutation on $[n]$ such that $Id_G(A_i) = Id_G(B_{\tau(i)})$.

□

The function τ in part (1) is not injective in general, but if we duplicate B enough times we can find a new function $\tilde{\tau}$ from the simple components of A to the simple components of B^N, for N large enough, such that $\tilde{\tau}$ is injective. If $Id_G(B_j) \subseteq Id_G(A_i)$ for B_j, A_i simple means that there is a graded embedding $A_i \hookrightarrow_G B_j$ then we can find a graded embedding $A \hookrightarrow_G B^N$ for N large enough, so we just proved that

Theorem 28. *Let $A_1, ..., A_n$ and $B_1, ..., B_m$ be G-simple algebras and define $A = \bigoplus_1^n A_i, \quad B = \bigoplus_1^m B_j$*

1. *Suppose that whenever $Id_G(B_j) \subseteq Id_G(A_i)$ than there is a graded embedding $A_i \hookrightarrow_G B_j$. If $Id_G(B) \subseteq Id_G(A)$ then there is a graded embedding $A \hookrightarrow_G B^N$ for N large enough.*

2. *Suppose that whenever $Id_G(B_j) = Id_G(A_i)$ then there is a graded isomorphism $A_i \cong_G B_j$. If the sets $\{A_i\}_1^n$ and $\{B_j\}_1^m$ are minimal and $Id_G(A) = Id_G(B)$ then $m = n$, and there is a permutation $\tau \in S_n$ such that $A_i \cong_G B_{\tau(i)}$, and in particular $A \cong_G B$.*

For example, part (1) is true for G abelian, A_i, B_j are finite dimensional and $\mathbb{K}$ is algebraically closed with $char(\mathbb{K}) = 0$. Part (2) is also true when the conditions in one of [2, 1] is true. Notice that by our construction, if A is simple then we can always choose $N = 1$ in part (1).

We continue to the next generalization where B can be any finite dimensional algebra, and we want to reduce our problem to the semisimple case.

We first recall that in the non-graded case, if B is Artinian (and in particular if B is finite dimensional) then its Jacobson radical J is nilpotent and if $J=0$ then B is a direct sum of simple algebras ([9]).

Let J be the Jacobson radical of B and $\pi : B \to {}^{B}/_{J}$ the projection map, then $B_{ss} = {}^{B}/_{J}$ is semi simple, and therefore a direct sum of simple algebras. In general we only know that $Id(B) \subseteq Id(B_{ss})$ so if $Id(B) \subseteq Id(A)$, then we don't know if there is any relation between $Id(A)$ and $Id(B_{ss})$. Assume that we can show that we actually have $Id(B) \subseteq Id(B_{ss}) \subseteq Id(A)$ so in a sense we don't add too much identities when we move from $Id(B)$ to $Id(B_{ss})$.

Wedderburn Malcev theorem states that we can write $B = B_{ss} \oplus J$ where the direct sum is as vector spaces, so the idea is to show that

$$Id(B) \subseteq Id(A) \;\Rightarrow\; Id(B_{ss}) \subseteq Id(A) \;\Rightarrow\; A \hookrightarrow B_{ss} \;\Rightarrow\; A \hookrightarrow B$$

A similar process can be done in the graded case. Define J_G to be the graded Jacobson radical of B, so J_G is the intersection of all maximal right graded ideals of B. Assume now that G is a finite group and that $|G|^{-1} \in B$. In [10] (theorem 4.4), Cohen and Montgomery proved that this condition implies that the graded Jacobson radical J_G equals to the non-graded radical J. The result about the Wedderburn Malcev decomposition $B \cong B_{ss} \oplus J_G$ can be found in [11] (corollary 2.8). It is also well known that if $J_G = \{0\}$ then B is a finite direct product of G-simple algebras.

Before we continue, we remark that some of these properties are true in a more general setting.

Lemma 29. *Assume that G is a finite group. Let A be G-semisimple algebra, and B a finite dimensional G-graded algebra such that $Id_G(B) \subseteq Id_G(A)$ and $|G|^{-1} \in B$. Denote by J_G the graded Jacobson radical of B and $B_{ss} = {}^{B}/_{J_G}$ its graded semi simple image, then $Id_G(B_{ss}) \subseteq Id_G(A)$.*

Proof. Assume by negation that there is $f \in Id_G(B_{ss}) \backslash Id_G(A)$, wlog homogeneous. B is finite dimensional so J is nilpotent, and because $|G|^{-1} \in B$ then $J_G = J$ and we can find $k \in \mathbb{N}$ such that $(J_G)^k = 0$. $Id_G(B_{ss})$ is an ideal so $h_i \cdot f \in Id_G(B_{ss})$ for any homogeneous polynomial h_i. Since $B_{ss} \cong {}^{B}/_{J_G}$ then $h_i \cdot f$ sends any graded assignment in B to J_G.

From Lemma 24, since $f \notin Id_G(A)$ and A is G-simple, we can find h_i such that $\prod_1^k (h_i \cdot f) \notin Id_G(A)$, but for any assignment $\bar{b}$ in B we get that

$$\prod_1^k (h_i \cdot f)(\bar{b}) \in \prod_1^k J_G = 0 \quad \Rightarrow \quad \prod_1^k (h_i \cdot f) \in Id_G(B)$$

and we get a contradiction since $Id_G(B) \subseteq Id_G(A)$.

This shows that there are no f such that $f \in Id_G(B_{ss}) \backslash Id_G(A)$, or in other words $Id_G(B_{ss}) \subseteq Id_G(A)$. ∎

Now combine this lemma, the last theorem and Wedderburn-Malcev theorem for the graded case to get

Theorem 30. *Let G be a finite abelian group. Let A, B be two be G-graded algebras over algebraically closed field $\mathbb{K}$ with $char(\mathbb{K}) = 0$. Assume that A is G-semisimple, B has a a unit and $Id_G(B) \subseteq Id_G(A)$. Then there is a graded embedding $A \hookrightarrow_G B^N$ for N large enough, and if A is simple then we can choose $N = 1$.*

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